MATRICES

EXPLORING MATHEMATICS ON YOUR OWN 19

MATRICES

J M

A mass of information can be concisely presented in table form. A table is a matrix and *Matrices* explores how it is possible to use two matrices together to gain more information about a situation.

Situations considered are networks, costing, information, geometry and probability. The ideas and methods of using matrices are developed from road communications between towns—a network. The technique is continued in finding the cost of several items and in information sought from a table. Geometry is considered as transformations and transformations as matrices. The probability situations include finding the probability of a student studying maths on the last day of term when his study habits are known.

The general approach is to find an answer to the situation by conventional means and to discover how matrices can describe the situation and provide the answer more effectively, and how they can also give more information about the situation.

The applications of matrix methods are increasing and this book gives a beginning and insight into the use and power of these methods.

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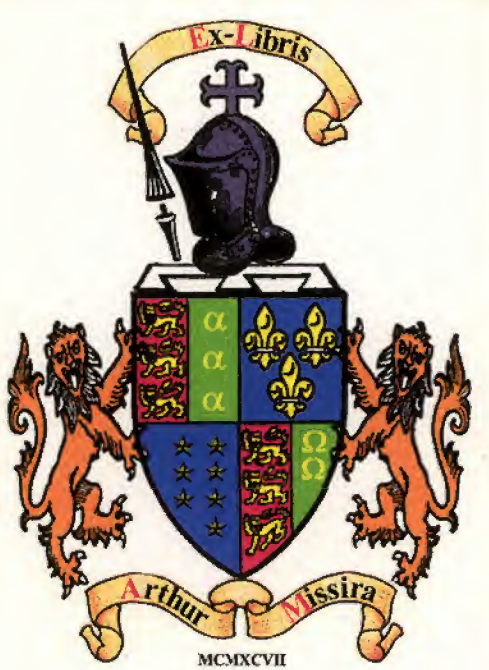

EXPLORING MATHEMATICS ON YOUR OWN

Matrices

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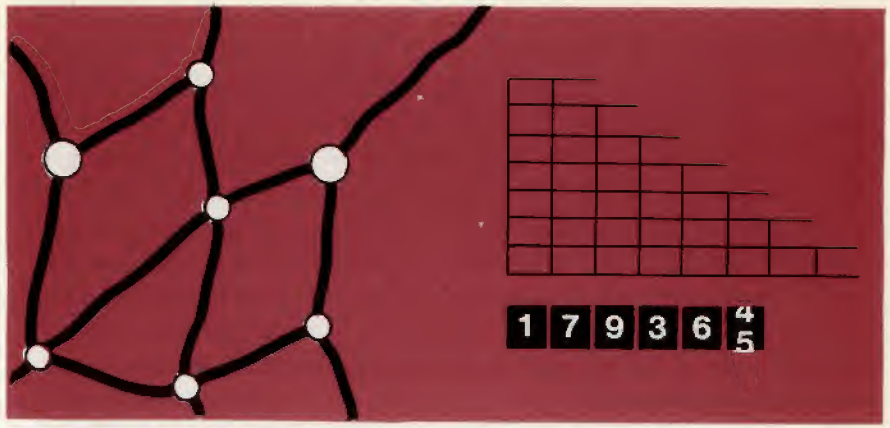


The matrix

Tables of information

We are often using tables—bus timetables, duty rotas, income tax tables. A table showing distances in miles between towns is like this:

London								
498	Aberdeen							
111	411	Birmingham						
54	552	160	Brighton					
115	490	88	134	Bristol				
74	563	185	82	189	Dover			
193	314	110	246	196	257	Leeds		
184	333	81	237	162	257	41	Manchester	
56	473	63	97	70	129	164	144	Oxford

If we want to know the distance from Aberdeen to Bristol we look down the Aberdeen column until we reach the Bristol row and find the distance to be 490 miles. What is the distance from Brighton to Oxford?, from Manchester to London?

Tables are a concise method of presenting a mass of information. When we construct a table from a collection of data, we generally arrange the data in rows and columns. We extract information from the table by reading the entry corresponding to a row and column intersection.

The matrix

Any table is a matrix. A matrix is a compact store of information, given by numbers arranged in rows and columns.

Besides the information which a matrix itself contains there are situations in which the matrix as a whole is used with another matrix. Generally we make a single matrix from a collection of data and we are content with the conciseness of the matrix in presenting the data and the efficient way we have of obtaining information from the matrix. There are several instances where it is possible to gain more information about a situation if we can use two matrices together. The exploration of combining two matrices has led to many interesting and useful results. A map of towns and their connection by road provides a situation in which we can explore using two matrices together. In this chapter we shall see first how a single matrix is formed.

Four towns a, b, c and d are connected by road as shown in Fig. 1.

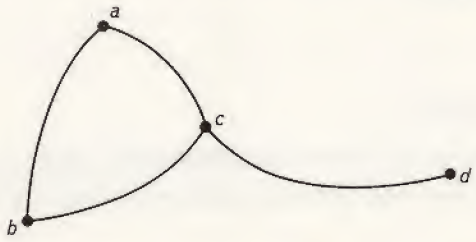


Fig. 1

Any situation which can be represented by nodes (the towns) and links (the roads) is termed a *network*.

Matrices to describe networks

We first have to find a way of representing the network of Fig. 1 by a matrix. The information we wish to portray is whether two towns are connected by a road. We begin our matrix by putting the towns at the heads of the columns and at the beginnings of the rows to form a table, as:

	a	b	c	d
a				
b				
c				
d				

To fill in the table we put a 1 entry if the towns are connected

by a road and an 0 if they are not. Town a has no road to itself, so the first entry in row a, column a is 0.

	a	b	c	d
a	0			
b				
c				
d				

Town a is connected by a road to b and to c but not to d; so the first row of the table is:

	a	b	c	d
a	0	1	1	0
b				
c				
d				

The complete table for the network of Fig. 1 is:

	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0.

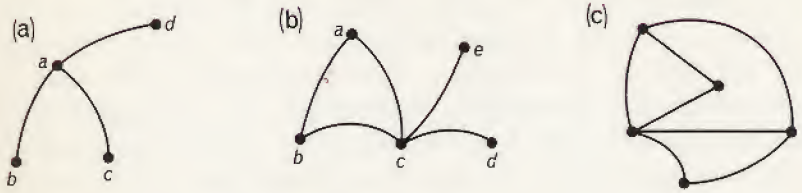
We now rewrite the table in round brackets, to keep its entries together, call it a matrix and denote it by a capital letter in bold type. The matrix for the network of Fig. 1 is

$$\mathbf{M} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

The importance of the matrix is that it has changed a geometrical diagram into an arithmetical table and we shall now be able to work with numbers.

EXERCISE SET 1

1. Find the matrices to describe the networks.



2. Draw the networks described by the matrices.

(a)
$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

(b)
$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array}\begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(c)
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

3. The sum of the entries in row c of the matrix **M** describing the network of Fig. 1 is 3. What does the 3 mean?

4. The *leading diagonal* of a matrix is from the top left to bottom right of a matrix. Why are the matrices describing these networks symmetrical about the leading diagonal?

Extending the use of a network matrix

What would be the matrix to describe the network of Fig. 2?

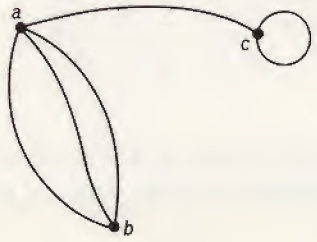


Fig. 2

There are three roads between a and b. The choice of 0 to describe no road connecting the towns and 1 if the towns are joined by one road extends to allow us to use the number of roads connecting one town to another, so here a to b has the entry 3. c is connected to itself in two ways, either way round the ring road, so the entry c to c is 2. The matrix to describe the network of Fig. 2 is:

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} \end{array}.$$

Can we describe a network which has some 'one-way' restrictions in it, such as the network of Fig. 3?

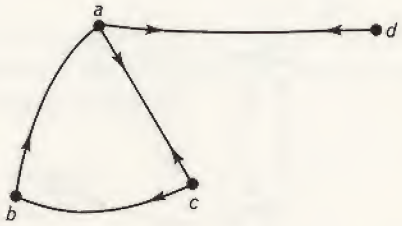


Fig. 3

Now we shall only be able to travel between two towns if there is a road joining them and the arrow says we may go that way. From town a we can travel to c and d, but not to b; so the first row of the matrix will be:

$$\begin{array}{c} \\ \text{from} \quad a \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \quad \text{to} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Note the "from—to" convention we need to make for writing the entries; from a to b is not possible, but from b to a is.

The complete matrix to describe the network of Fig. 3 is:

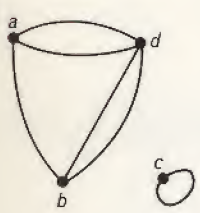
$$\begin{array}{c} \\ a \\ b \\ c \\ \text{from} \quad d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \quad \text{to} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

EXERCISE SET 2

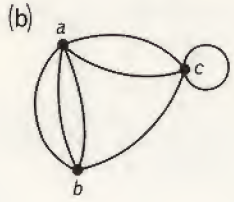
1. The matrix of Fig. 3 is *not* symmetrical about the leading diagonal—why? The sum of the entries in row a of this matrix is 2—what does the 2 mean? The sum of the entries in column a is 3—what does the 3 mean?

2. Find the matrices to describe the networks:

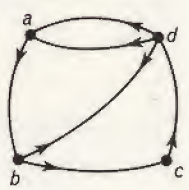
(a)



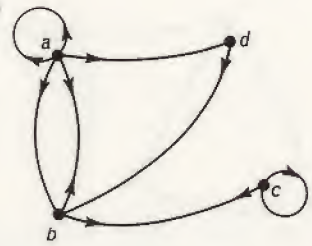
(b)



(c)



(d)



3. Draw the networks corresponding to the matrices.

(a)

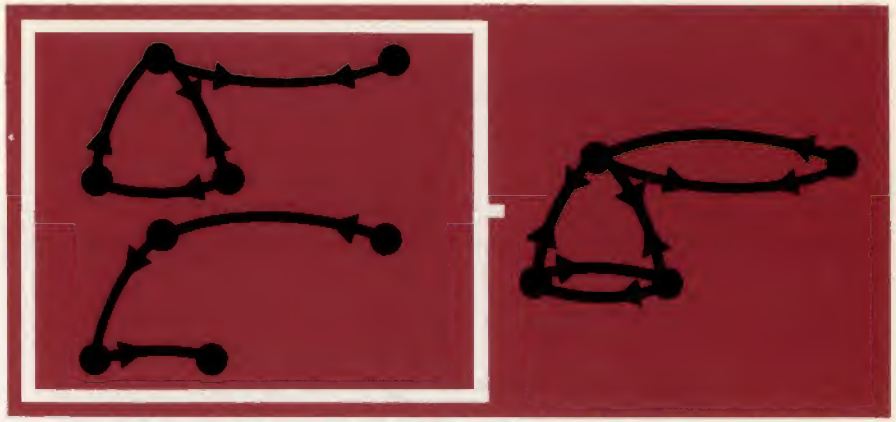
	a	b	c
a	0	0	3
b	0	0	1
c	3	1	0

(b)

	a	b	c
a	0	2	0
b	2	0	0
c	0	0	2

(c)

	a	b	c	d
a	0	1	2	0
b	1	1	0	0
c	1	0	0	1
d	1	0	0	2

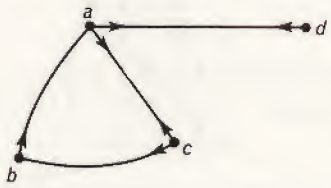


Combining matrices

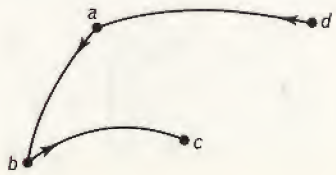
A matrix is able to describe route connections within a network. We shall now explore if we can use two matrices together to give more information about networks.

Adding two matrices

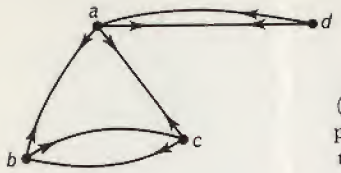
The route diagram of Fig. 3 was



New routes have been allowed and constructed, described by the diagram

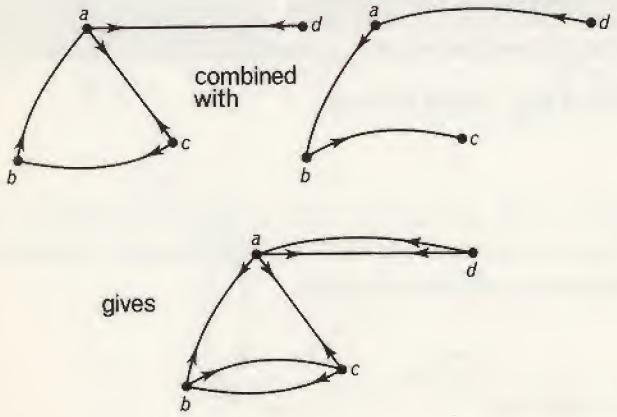


so the present situation is



(Routes between b and c represent separate one-way systems.)

That is



The matrices for these networks are:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{combined with} \quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{gives} \quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

How are the first two matrices combined to give the resultant matrix? The arithmetical process of obtaining the resultant matrix is to *add* corresponding entries in the other matrices. We call this method of combining matrices *addition*, use the symbol $+$ and write:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

EXERCISE SET 3

1. Add the matrices and draw the corresponding networks.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

2. The existing routes between four towns is described by the matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is proposed to make further connections such that each town is connected to every other town by at least one route. What is the matrix which describes a network such that each town has a route to every other town? From the matrices, form the matrix to describe the extra routes needed.

Multiplying two matrices

Towns a, b, c and d have a train service between them described by the diagram of Fig. 4.

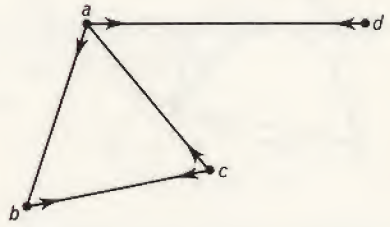


Fig. 4

The matrix **T** which describes this service is:

$$\mathbf{T} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{l} \text{to} \\ \\ \\ \\ \text{from} \end{array}.$$

Also between these four towns there is a bus service described by the diagram of Fig. 5.

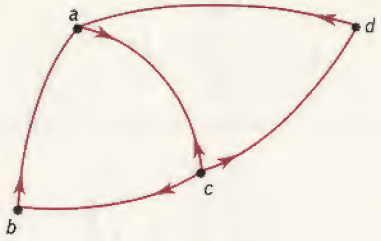


Fig. 5

The matrix **B** which describes this service is:

$$\mathbf{B} = \begin{array}{c} \\ a \\ b \\ c \\ \text{from } d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \text{ to} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{array}$$

The complete service can be put on one diagram, Fig. 6.

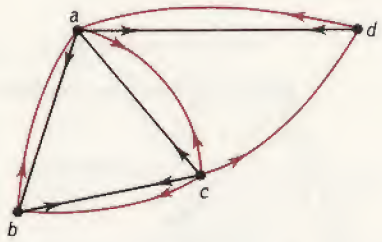


Fig. 6

I want to take a journey on the train and then on the bus, so that from one town I go to another town by train and from this town I take a bus to some other town. Between which towns is it possible to take this double journey of a train ride followed by a bus ride? From Fig. 6 I see I may go by train from *a* to *b* or *d*. From *b* I can go by bus back to *a* and from *d* I can go by bus back again to *a*. So I may take a train journey followed by a bus journey from *a* to *a* twice (via *b* or *d*) and not from *a* to any other town. If I lived at *b* I can go by train to *c* (only) and from *c* I can take a bus back to *b*,

or to *a* or to *d*; so a train ride followed by a bus ride is possible between *b* and *a*, *b* or *d*. Using Fig. 6 check that the possible journeys of a train ride followed by a bus ride between the towns is described by the matrix:

$$\begin{array}{c} \\ a \\ b \\ c \\ \text{from } d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \text{ to} \\ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{array}$$

Using matrices in this situation we have a train matrix **T** which describes possible journeys between towns by train, a bus matrix **B** describing possible bus journeys between the towns and a matrix made by answering the question "What journeys are possible between towns if I take a train ride followed by a bus ride?" Are we able to put the two matrices **T** and **B** together to give the final matrix? That is, can we find a way of saying the following?

$$\mathbf{T} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \text{ combined with } \mathbf{B} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \text{ gives } \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{array}$$

We can find the method we have to use by looking at the information given by the matrices **T** and **B**.

In **T** the *row a* is

T	*a*	*b*	*c*	*d*
a	0	1	0	1

which says that I can take a train *from a* and get to *b* or *d*.

In **B**, the *column a* is

B	*a*
a	0
b	1
c	1
d	1

which says that I can get *to a* by bus from *b*, *c* or *d*.

Now I am at b or d by train and from b or d I can go by bus to a, so I can go two ways from a to a by taking a train ride and then a bus ride. Note that journeys are possible where the 1's in the *row a* of **T** correspond with the 1's in the *column a* of **B** and we can do the arithmetic by saying

T	a	b	c	d
a	0	1	0	1

with

B	a
a	0
b	1
c	1
d	1

gives

$$0 \times 0 + 1 \times 1 + 0 \times 1 + 1 \times 1 = 2$$

where the entries in the row are multiplied by the *corresponding* entries in the column and then *added*. Is this process satisfactory for finding the number of possible routes for a train ride followed by a bus ride from a to b?—we know the answer is none. The *row a* in **T** is

T	a	b	c	d
a	0	1	0	1

and the *column b* in **B** is

B	b
a	0
b	0
c	1
d	0.

The number of possible routes is therefore

$$0 \times 0 + 1 \times 0 + 0 \times 1 + 1 \times 0 = 0.$$

This process gives the correct result each time since it picks up where the 1's, showing that a train or bus route exists between the towns, coincide. We have done a to a, a to b and for a to c we would work

T	a	b	c	d
a	0	1	0	1
b	0	0	1	0
c	1	1	0	0
d	1	0	0	0

with

B	a	b	c	d
a	0	0	1	0
b	1	0	0	0
c	1	1	0	1
d	1	0	0	0

gives

	a	b	c	d
a			$0 \times 1 + 1 \times 0 + 0 \times 0 + 1 \times 0 = 0$	
b				
c				
d				

Each entry in the final matrix is obtained by taking each row and column in turn, multiplying corresponding entries and adding, and then placing the result in the same row and column position of the final matrix.

For two smaller matrices this process would be worked as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ gives } \begin{pmatrix} 0 \times 1 + 1 \times 1 = 1 & \\ & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ gives } \begin{pmatrix} 1 & 0 \times 1 + 1 \times 0 = 0 \\ & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ gives } \begin{pmatrix} 1 & 0 \\ 1 \times 1 + 0 \times 1 = 1 & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ gives } \begin{pmatrix} 1 & 0 \\ 1 & 1 \times 1 + 0 \times 0 = 1 \end{pmatrix},$$

that is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ gives $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

This process of row on column multiplication of corresponding entries and then their addition is called *multiplication* of matrices and the symbol $\times$ is used (or omitted!) between the matrices and we write

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Continue to check, by matrix multiplication, the entries in the matrix for a train ride followed by a bus ride.

EXERCISE SET 4

1. (a) Draw the network for train routes as described by the matrix $\mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and bus routes as described by the matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ between two towns a and b. Write down from the network the possible routes between towns for a train ride followed by a bus ride—the result should be the matrix we computed above.

(b) Compute the matrix for $\mathbf{BT} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

What journeys within the network does this matrix describe?
(Note that the result of **TB** is *not* the same as the result of **BT**; that is **TB** $\neq$**BT** and in general multiplication of matrices is not commutative*).

(c) Compute the matrix for $\mathbf{T}^2 = \mathbf{TT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

What journeys within the network does this matrix describe?

(d) Compute the matrix for **TBT**. What journeys within the network does this matrix describe? (Note that the result of **TBT** can be computed by first finding the result of **TB** and post-multiplying this by **T** as (**TB**)**T** or by computing **BT** and pre-multiplying this result by **T** as **T**(**BT**). These two ways of computing **TBT** give the same result and it can be shown that multiplication of matrices is associative*).

2. Compute the resultant matrix in the following cases:

(a) $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}^2$

(e) $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\right]$ and $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

The example (e) illustrates that matrix multiplication is distributive* over matrix addition).

* For an explanation of these terms and their use in mathematics see *Finite Mathematical Systems*, pp. 8–13, Number 17 in this series.

3. The bus service between three villages a, b and c is described by the network

I enjoy walking and have found that the most enjoyable walking routes between the villages is as described by the diagram

(a) Find the bus (**B**) and walking (**W**) matrices for these routes.

(b) Between which villages is it possible for me to take a bus ride followed by a walk?

(c) Between which villages is it possible to take a walk followed by a bus ride?

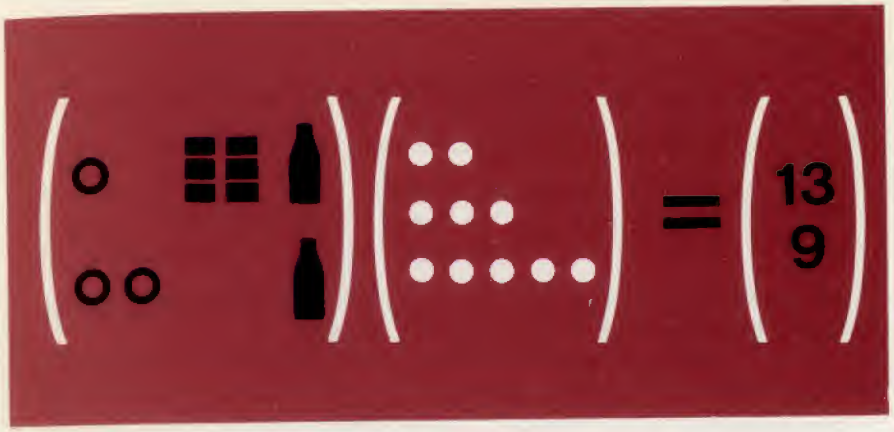
(d) On a sunny day (and when feeling energetic) I take walks between the villages. Between which villages can I take two walks?

(e) Better than two consecutive walks would be a walk followed by a bus ride followed by a walk. Between which villages can I take a journey such as this?

(f) On a rainy day, between which villages can I take three bus rides?

The effectiveness of the matrix method

The arithmetic process devised for multiplying matrices enables the existence and number of routes between two places to be *calculated*, quickly and efficiently. It presents a better process than trying to follow routes around a diagram. The usefulness of this method also lies in the existence of many other situations in which the same method can be applied.



Matrices for costing

Andrew and Brenda visit the local snack bar. Andrew has a doughnut, two bars of chocolate and a glass of milk. A doughnut costs 2p, the chocolate bar 3p and a glass of milk 5p. So Andrew has to pay $1 \times 2 + 2 \times 3 + 1 \times 5 = 13$p.

We can find what Andrew pays by using matrices and matrix multiplication, since in matrix multiplication what we do is to multiply numbers and then add them. The matrix which describes the items Andrew buys is

$$\begin{array}{c} \\ \text{Andrew} \end{array}\begin{array}{c} \begin{array}{ccc} \text{doughnut} & \text{chocolate bar} & \text{milk} \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \end{array} = \mathbf{m} \text{ say.}$$

A matrix which has one row or one column is known as a *row vector** or a *column vector* and is denoted by a small letter in heavy type. **m** is a row vector. The price matrix is

$$\begin{array}{l} \\ \text{doughnut} \\ \text{chocolate bar} \\ \text{milk} \end{array}\begin{array}{c} \text{price} \\ \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \end{array} = \mathbf{n} \text{ say and } \mathbf{n} \text{ is a column vector.}$$

* In *Basic Concepts of Vectors*, pp. 43–45, Number 16 in this series, the addition of two vectors is considered. Corresponding entries are added in vector addition and as this is how we would add the matrices $(1\ 2) + (3\ 4) = (4\ 6)$ we may link row and column matrices with vectors.

Then the amount Andrew pays is

$$\mathbf{mn} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = 1 \times 2 + 2 \times 3 + 1 \times 5 = 13\text{p.}$$

If Brenda has two doughnuts and a glass of milk, we can put the purchases of Andrew and Brenda in one matrix **P**, as

$$\begin{array}{c} \\ \text{Andrew} \\ \text{Brenda} \end{array}\begin{array}{c} \begin{array}{ccc} \text{doughnut} & \text{chocolate bar} & \text{milk} \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \end{array}$$

and can find the cost of their purchases by using the price vector **n**.

$$\begin{array}{c} \\ \text{Andrew} \\ \text{Brenda} \end{array}\begin{array}{c} \begin{array}{ccc} \text{doughnut} & \text{chocolate bar} & \text{milk} \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \end{array}\begin{array}{l} \text{price} \\ \text{doughnut} \\ \text{chocolate bar} \\ \text{milk} \end{array}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$= \begin{array}{c} \\ \text{Andrew} \\ \text{Brenda} \end{array}\begin{array}{c} \text{price} \\ \begin{pmatrix} 1 \times 2 + 2 \times 3 + 1 \times 5 \\ 2 \times 2 + 0 \times 3 + 1 \times 5 \end{pmatrix} \end{array} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}.$$

When using matrices in this way it is important to place the items correctly in the matrices and put the matrices together so that the items in the *columns* of the first matrix are the *rows* of the following matrix.

On another visit to the snack bar, Andrew had a doughnut and a glass of milk, Brenda had two bars of chocolate and two glasses of milk. What was the cost of their purchases?

In the section on networks the matrices were square—they had as many rows as columns, since we used the towns as headings of columns and beginnings of rows. The first purchase matrix was of the form

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

which has two rows and three columns—a *rectangular* matrix. Such a matrix is described as a "two by three" matrix (shorthand 2×3), the number of rows being given first.

The row vector $\mathbf{m} = (1 \quad 2 \quad 1)$ is a 1×3 matrix and the price vector $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ is a 3×1 matrix.

Multiplying rectangular matrices

Matrix multiplication of rectangular matrices is performed as follows:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1\times 2 + 2\times 3 + 1\times 5 = 13 & \\ & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 1\times 1 + 2\times 2 + 1\times 3 = 8 \\ & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 2\times 2 + 0\times 3 + 1\times 5 = 9 & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 9 & 2\times 1 + 0\times 2 + 1\times 3 = 5 \end{pmatrix}$$

so

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 9 & 5 \end{pmatrix}.$$

Note that a 2×3 matrix multiplied by a 3×2 matrix has given a 2×2 matrix.

EXERCISE SET 5

Multiply (if possible) the following matrices

(a) $\begin{pmatrix} 0 & 1 & 3 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix}$

(e) $(1 \quad 2 \quad 3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(f) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(1 \quad 2 \quad 3)$

(g) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & 0 \\ 6 & 6 & 4 \end{pmatrix}\begin{pmatrix} 10 \\ 20 \\ 14 \end{pmatrix}$

The cost of wedding cakes

A confectioner bakes three different kinds of Wedding Cake. The quantities in each cake in convenient units is described by the ingredient matrix **I**.

	I	fruit	butter	sugar	flour	eggs	brandy
Cake	*A*	2	8	8	$\frac{3}{4}$	5	3
	B	$1\frac{1}{2}$	6	6	$\frac{1}{2}$	4	2
	C	1	4	4	$\frac{1}{4}$	3	1

In one week the confectioner has orders for two of *A*, four of *B* and three of *C*. The price of the ingredients in new pence per unit is described by the price vector **P**.

P	fruit	butter	sugar	flour	eggs	brandy
price	16	1	$\frac{1}{2}$	4	$1\frac{1}{2}$	20

What is the cost of making the cakes?
The order vector, **o** is

o	*A*	*B*	*C*
order	2	4	3

and this vector with the ingredient matrix will give the amount of ingredients required to fulfil the order.

$$\begin{array}{rl} & \begin{array}{ccc} A & B & C \end{array} \\ \mathbf{oI} = \text{order}(& 2 \quad 4 \quad 3) \end{array} \begin{array}{c} \\ \begin{matrix} A \\ B \\ C \end{matrix} \end{array} \begin{pmatrix} \text{fruit} & \text{butter} & \text{sugar} & \text{flour} & \text{eggs} & \text{brandy} \\ 2 & 8 & 8 & \frac{3}{4} & 5 & 3 \\ 1\frac{1}{2} & 6 & 6 & \frac{1}{2} & 4 & 2 \\ 1 & 4 & 4 & \frac{1}{4} & 3 & 1 \end{pmatrix}$$

$$= \text{order}(\begin{array}{cccccc} \text{fruit} & \text{butter} & \text{sugar} & \text{flour} & \text{eggs} & \text{brandy} \\ 13 & 52 & 52 & 4\frac{1}{4} & 35 & 17 \end{array}).$$

The order vector and ingredient matrices have been arranged so that matrix multiplication will give the amount of ingredients required.

The total cost of the cakes is now found by using the price vector as

$$\begin{array}{c} \begin{array}{cccccc} \text{fruit} & \text{butter} & \text{sugar} & \text{flour} & \text{eggs} & \text{brandy} \end{array} \\ \text{order}(\begin{array}{cccccc} 13 & 52 & 52 & 4\frac{1}{4} & 35 & 17 \end{array}) \end{array} \begin{array}{c} \text{price} \\ \begin{matrix} \text{fruit} \\ \text{butter} \\ \text{sugar} \\ \text{flour} \\ \text{eggs} \\ \text{brandy} \end{matrix} \end{array} \begin{pmatrix} 16 \\ 1 \\ \frac{1}{2} \\ 4 \\ 1\frac{1}{2} \\ 20 \end{pmatrix}$$

$$= \text{order}\ (695\tfrac{1}{2}) = £6{\cdot}95\tfrac{1}{2}.$$

Once the information has been put into matrix form and the matrices arranged so that the column items of one matrix agree with the row items of the next matrix, matrix multiplication gives an effective and efficient method of finding the total cost.

The cost of a model car racing set

A model car racing track is made up in two packs. Pack No. 1 contains 4 straights and 2 180° curves and Pack No. 2 contains 6 straights, 6 180° curves and 4 90° curves. From these packs, three kits *A*, *B* and *C* are made by using Pack No. 1 for kit *A*, Pack No. 1 and Pack No. 2 for kit *B* and for kit *C* there are three of Pack No. 1 and two of Pack No. 2. In new pence a straight costs 10p, a 180° curve 20p and a 90° curve 14p. How much is each kit?

First put the information into matrices!

$$\begin{array}{c} \\ \begin{matrix} \text{Pack No. 1} \\ \text{Pack No. 2} \end{matrix} \end{array} \begin{pmatrix} \text{straights} & 180° & 90° \\ 4 & 2 & 0 \\ 6 & 6 & 4 \end{pmatrix}$$

$$\begin{array}{c} \\ \begin{matrix} A \\ B \\ C \end{matrix} \end{array} \begin{pmatrix} \text{Pack No. 1} & \text{Pack No. 2} \\ 1 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\text{price}(\begin{array}{ccc} \text{straight} & 180° & 90° \\ 10 & 20 & 14 \end{array}).$$

To find the cost of the kits *A*, *B* and *C* we put these matrices together as

$$\begin{array}{c} \begin{array}{cc} \text{Pack} & \text{Pack} \\ \text{No. 1} & \text{No. 2} \end{array} \\ \begin{matrix} A \\ B \\ C \end{matrix} \end{array} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix} \quad \begin{array}{c} \\ \begin{matrix} \text{Pack No. 1} \\ \text{Pack No. 2} \end{matrix} \end{array} \begin{pmatrix} \text{straight} & 180° & 90° \\ 4 & 2 & 0 \\ 6 & 6 & 4 \end{pmatrix} \quad \begin{array}{c} \text{price} \\ \begin{matrix} \text{straight} \\ 180° \\ 90° \end{matrix} \end{array} \begin{pmatrix} 10 \\ 20 \\ 14 \end{pmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} \text{straight} & 180° & 90° \end{array} \\ \begin{matrix} A \\ = B \\ C \end{matrix} \end{array} \begin{pmatrix} 4 & 2 & 0 \\ 10 & 8 & 4 \\ 24 & 18 & 8 \end{pmatrix} \quad \begin{array}{c} \text{price} \\ \begin{matrix} \text{straight} \\ 180° \\ 90° \end{matrix} \end{array} \begin{pmatrix} 10 \\ 20 \\ 14 \end{pmatrix} = \begin{array}{c} \text{price} \\ \begin{matrix} A \\ = B \\ C \end{matrix} \end{array} \begin{pmatrix} 80 \\ 316 \\ 712 \end{pmatrix}.$$

So kit A costs 80p, kit B £3·16 and kit C £7·12.

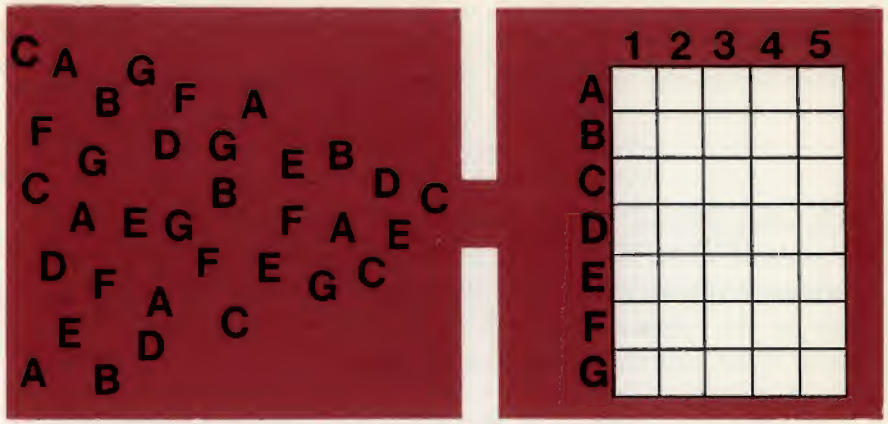
EXERCISE SET 6

1. A person orders plants from a nursery for himself and his neighbours. He orders 12 antirrhinums, 24 stocks, 24 sweet-williams, 12 chrysanthemums, 6 polyanthuses and 12 mesembryanthemums. One neighbour wants 24 antirrhinums, 12 asters, 12 chrysanthemums and 12 polyanthuses. The other neighbour requires 12 stocks, 12 sweet-williams, 12 asters and 12 mesembryanthemums. Half-a-dozen antirrhinums cost 12p, half-a-dozen stocks 10p, half-a-dozen sweet-williams 12p, half-a-dozen asters 12p, chrysanthemums are 5p each, polyanthuses 3p each and mesembryanthemums 10p a half-dozen. What is the cost of each order?

2. The manufacture of models containing electronic equipment consists of making three different parts into sub-assemblies and then

sub-assemblies into the model. The cost of the three parts, Part 1, Part 2 and Part 3, is 3, 2 and 4 units respectively. 3 of Part 1, 4 of Part 2 and 3 of Part 3 make sub-assembly 1; 2 of Part 1, 3 of Part 2 and 5 of Part 3 make sub-assembly 2. Model *A* consists of 3 of sub-assembly 1 and 3 of sub-assembly 2, Model *B* of 2 of sub-assembly 1 and 4 of sub-assembly 2 and Model *C* has 3 of sub-assembly 1 and 5 of sub-assembly 2. In one day 8 of Model *A* are made, 5 of model *B* and 4 of model *C*. What is the parts per model matrix and the cost of making the models for the day?

3. A firm of confectioners makes up three selection boxes of sweets for Christmas sales. The box "Sleigh Ride" has 2 wafer biscuits, a packet of whizzos, 3 bars of chocolate (one nut, one cream and one caramel centre), a packet of toffees, a packet of sherbert and 3 balloons. "Santa Claus" box has 1 wafer biscuit, a packet of whizzos, a bar of nut chocolate, a packet of toffees, a packet of sherbert, a pack of chocolate cigarettes, a chocolate pipe and 6 balloons. The box "Jingle Bells" contains a packet of whizzos, 4 bars of chocolate (one nut, two cream, one caramel centre), 2 packs of chocolate cigarettes, 2 chocolate pipes, 6 chocolate coins and 12 balloons. To the manufacturer a wafer biscuit costs 1p, a packet of whizzos 3p, a bar of chocolate (any centre) 2p, a packet of toffees 4p, a packet of sherbert 1p, a packet of chocolate cigarettes 2p, a chocolate pipe 1p, a chocolate coin 1p and a balloon $1\frac{1}{2}$p. The cost of packaging of each box is 5p. A shop orders 10 "Sleigh Ride", 15 "Santa Claus" and 10 "Jingle Bells". What is the cost of the order to the manufacturer?



Matrices for information

In a factory there are seven types of work and in each type there are five different salaries. The matrix, **M**, describing the number of workers in each type of work and salary is

Type of work

Type of Salary	**M**	*A*	*B*	*C*	*D*	*E*	*F*	*G*
	1	2	3	4	0	6	2	0
	2	4	2	6	2	8	4	0
	3	10	6	12	4	10	7	0
	4	3	0	5	1	2	3	1
	5	1	1	3	1	0	1	2

Let **n** be the row vector $(0\ 0\ 1\ 0\ 0)$

and **m** the column vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

What information is given by **nM**?

$$\mathbf{nM} = (0\ 0\ 1\ 0\ 0)\begin{pmatrix} 2 & 3 & 4 & 0 & 6 & 2 & 0 \\ 4 & 2 & 6 & 2 & 8 & 4 & 0 \\ 10 & 6 & 12 & 4 & 10 & 7 & 0 \\ 3 & 0 & 5 & 1 & 2 & 3 & 1 \\ 1 & 1 & 3 & 1 & 0 & 1 & 2 \end{pmatrix} = (10\ 6\ 12\ 4\ 10\ 7\ 0),$$

since, when the row vector **n** is matrix "multiplied" down each of the columns of **M**, the "1" in the vector **n** picks up the third entry down each column of **M**, and each result is put in the corresponding place to the column used. **nM** picks out the workers with type 3 salary. What information is given by **Mm**?

$$\mathbf{Mm} = \begin{pmatrix} 2 & 3 & 4 & 0 & 6 & 2 & 0 \\ 4 & 2 & 6 & 2 & 8 & 4 & 0 \\ 10 & 6 & 12 & 4 & 10 & 7 & 0 \\ 3 & 0 & 5 & 1 & 2 & 3 & 1 \\ 1 & 1 & 3 & 1 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 1 \\ 1 \end{pmatrix}.$$

As each row of **M** is matrix multiplied down the vector **m**, the "1" again picks up the entry in the fourth column of **M**. **Mm** picks out the type *D* workers. **nMm** will pick out the number of type *D* workers with type 3 salary.

Let $\mathbf{p} = (1\ 1\ 1\ 1\ 1)$ and $\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

What information is given by **pM**?

$$\mathbf{pM} = (1\ 1\ 1\ 1\ 1)\begin{pmatrix} 2 & 3 & 4 & 0 & 6 & 2 & 0 \\ 4 & 2 & 6 & 2 & 8 & 4 & 0 \\ 10 & 6 & 12 & 4 & 10 & 7 & 0 \\ 3 & 0 & 5 & 1 & 2 & 3 & 1 \\ 1 & 1 & 3 & 1 & 0 & 1 & 2 \end{pmatrix} = (20\ 12\ 30\ 8\ 26\ 17\ 3)$$

giving the total number of each type of worker. What will **Mq** and **pMq** give?

EXERCISE SET 7

1. Use appropriate vectors to find the following information from the matrix **M**.

(a) Workers of each type with type 1 or 2 salary.
(b) Type *E* and *F* workers for each type of salary.
(c) The total of type *B*, *C*, *D* and *E* workers with type 2, 3 or 4 salary.
(d) Total number of workers with more than type 3 salary.

2. Boys and girls in a class are classified according to the number of brothers and sisters they have by the matrix, **N**.

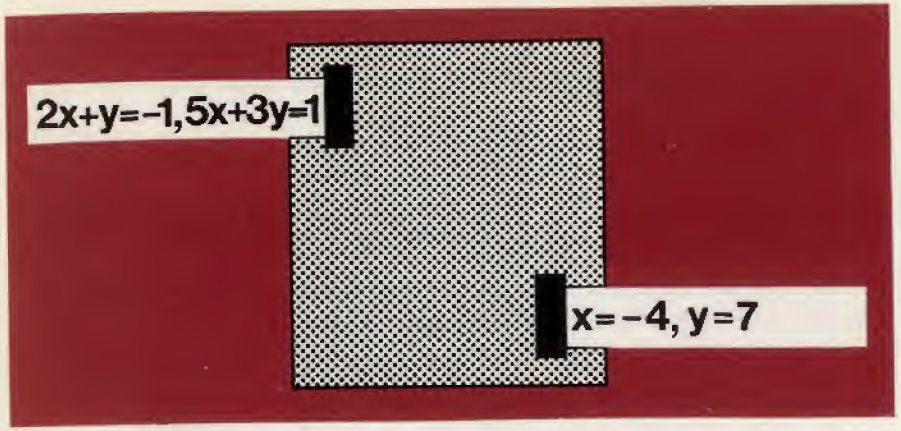
N

No. of sisters \ No. of brothers	0	1	2	3	4
0	8	3	0	1	0
1	0	5	4	0	1
2	3	2	1	0	0
3	0	1	0	0	0
4	1	0	0	0	0

Let $\mathbf{m} = (0\ 1\ 0\ 0\ 0)$; $\mathbf{p} = (1\ 1\ 1\ 1\ 1)$; $\mathbf{r} = (0\ 1\ 2\ 3\ 4)$;

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{s} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

(a) What information is given by **mN**, **Nn** and **mNn**?
(b) What choice of matrices would provide the information of
  (i) total number of members in the class to each classification of brothers
  (ii) total number of members in the class to each classification of sisters
  (iii) total number of members of the class?
(c) What information is given by the matrices **rN** and **Ns**?
(d) What information is given by the matrices **rNq**/**pNq**?
(e) What matrices would give the average number of brothers possessed by members of the class?



Matrices in the solution of linear equations

Matrices for linear equations

When we solve equations the important items in the equations are the numerical values, since it is these values which decide the answer. In the simultaneous linear equations

$$\begin{aligned} 2x + \ y &= -1 \\ 5x + 3y &= \ \ 1 \end{aligned}$$

the answer values for x and y (they could have been a and b or any other symbols) depend on the numerical values in the equations. We may write the values in the left-hand side of these equations as the matrix $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$—these values being those in the position of the coefficients of x and y. Writing x and y as the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and putting with the matrix as $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ we use matrix multiplication to find

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + \ y \\ 5x + 3y \end{pmatrix}$$

a matrix of the left-hand side of the equations.

The right-hand values in the equations we can write as the column vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

If we now write

$$\begin{pmatrix} 2x + \ y \\ 5x + 3y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

we have two matrices (2×1 column vectors) which are equal. Two matrices are equal only when their corresponding entries are equal; so

$$\begin{aligned} 2x + \ y &= -1 \\ 5x + 3y &= \ \ 1. \end{aligned}$$

We are now able to write the simultaneous linear equations

$$\begin{aligned} 2x + \ y &= -1 \\ 5x + 3y &= \ \ 1 \end{aligned}$$

in the matrix form

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

EXERCISE SET 8

1. Between which of the following sets of matrices is the equal sign justified?

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 0 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

2. If $\begin{pmatrix} a & 2b \\ 3c & -d \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 6 & 2 \end{pmatrix}$ what are the values of a, b, c and d?

3. Put the following sets of simultaneous equations into matrix form

(a) $\begin{aligned} 2x + 7y &= 3 \\ x + 4y &= 1 \end{aligned}$ (b) $\begin{aligned} 3a - 2b &= 14 \\ a - \ b &= \ 6 \end{aligned}$

(c) $\begin{aligned} 3P + 2Q &= \ \ 14 \\ 4P + 5Q &= -11 \end{aligned}$

Solving linear equations

To solve the equation $ax = b$, we divide both sides by a; we can think of this as *multiplying* both sides by $\frac{1}{a}$. We choose $\frac{1}{a}$ as $\frac{1}{a} \times a = 1$.

In the equation $ax = b$, multiplying both sides by $\frac{1}{a}$ gives

$\frac{1}{a} \times ax = \frac{1}{a} \times b$ which leads to $1x = \frac{b}{a}$ by using the associative law to write $\frac{1}{a} \times ax$ as $\left(\frac{1}{a} \times a\right)x$. As $1x = x$, we now obtain the solution for x as $\frac{b}{a}$. The two essentials chosen in this process of solution are: (i) $1x = x$, that 1 has the property that when multiplied by any value it leaves the value unchanged; and (ii) that $\frac{1}{a} \times a = 1$. For multiplication of numbers, 1 is known as the *identity element** and $\frac{1}{a}$ the *inverse** of a, normally denoted by a^{-1}.

The matrix method of solving equations

To solve the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

or $\mathbf{Ax} = \mathbf{h}$ where $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $\mathbf{x}$ and $\mathbf{h}$ are respectively the column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, we can explore to see if we can follow the same procedure as we have done to solve the equation $ax = b$. Firstly, we look for a matrix which when multiplied by

* See *Finite Mathematical Systems*, pp. 12–13, Number 17 in this series.

another matrix leaves this matrix unchanged. The matrix will be the *identity matrix*. Then we must find a matrix such that when it is multiplied by $\mathbf{A}$ it gives the identity matrix. This matrix would be the *inverse matrix*. If $\mathbf{I}$ is the identity matrix and $\mathbf{A}^{-1}$ the inverse of $\mathbf{A}$, then $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$.

So, to solve the matrix equation $\mathbf{Ax} = \mathbf{h}$ we would work

$$\begin{aligned} \mathbf{Ax} &= \mathbf{h} \\ \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{h} \text{ multiplying both sides by } \mathbf{A}^{-1} \\ \mathbf{Ix} &= \mathbf{A}^{-1}\mathbf{h} \text{ as } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{h} \text{ as } \mathbf{Ix} = \mathbf{x} \end{aligned}$$

But first we need to find the identity matrix ($\mathbf{I}$) and the inverse matrix ($\mathbf{A}^{-1}$).

The identity matrix

Multiplying a matrix by the identity matrix to give the same matrix is like multiplying a number by 1 to give the same number. The identity matrix must therefore be the same for all similar types of matrix.

If $\mathbf{Ix} = \mathbf{x}$ or $\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a 2×2 matrix which has 1's in the leading diagonal and 0's elsewhere.

What would be the identity matrix $\mathbf{I}$, when $\mathbf{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$?

The inverse of a matrix

If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ then the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$, is a matrix such that

$$\mathbf{A}^{-1}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\mathbf{A}^{-1}$ must be a 2×2 matrix since it is to be multiplied by a 2×2 matrix to give a 2×2 matrix.

We find $\mathbf{A}^{-1}$ by using a series of matrices which systematically change the matrix $\mathbf{A}$ into the identity matrix.

Let $\mathbf{E}_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{E}_1\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 5 & 3 \end{pmatrix}$ and there is a 1 in the top left-hand corner—the start of the identity matrix. Note that the form of $\mathbf{E}_1$ is the identity matrix with the reciprocal of the first entry of $\mathbf{A}$ as its first entry.

Let $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$, then $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ and there is now an 0 in the bottom left-hand corner.

$\mathbf{E}_2$ is the identity matrix but with minus the entry in the bottom left-hand corner of the $(\mathbf{E}_1\mathbf{A})$ matrix as its bottom left-hand entry.

Let $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $\mathbf{E}_3(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ and there is a 1 in the bottom right-hand corner. $\mathbf{E}_3$ is the identity matrix with the reciprocal of the bottom right-hand entry of the $(\mathbf{E}_2\mathbf{E}_1\mathbf{A})$ matrix as its bottom right-hand entry.

Let $\mathbf{E}_4 = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$, then $\mathbf{E}_4(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$—the identity matrix. $\mathbf{E}_4$ is again the identity matrix but with minus the top right-hand entry of the $(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A})$ matrix as its top right-hand entry.

Now we have $\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$, so as $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, $\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \mathbf{A}^{-1}$.

$$\begin{aligned}\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 &= \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{5}{2} & 1 \end{pmatrix} \text{ multiplying the matrices in pairs} \\ &= \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.\end{aligned}$$

We check that in fact $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ does equal $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The process for finding the inverse of a matrix is cumbersome but systematic. When the number of unknowns is large, the process is extended and the finding of the inverse can be programmed for a computer. The following exercises illustrate how a general formula can be established for finding the inverse of a matrix.

EXERCISE SET 9

1. Using suitable **E** matrices find the inverse of each of the matrices.

(a) $\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

Can you see a pattern between the matrix and its inverse? Write down the inverse of the matrix $\begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}$. Check your answer.

Write down the inverse of the matrix $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. Check your answer. (If your written answer is wrong, find the correct inverse using the **E** matrices —where has the dividing factor 10 come from?)

2. By using suitable **E** matrices show that the inverse of the general matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \begin{pmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}.$$

Since *each* entry in the inverse has the factor $\frac{1}{ad-bc}$, we can simplify the matrix by extracting this factor and writing

$$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \text{ as } \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

This gives us a general formula for finding the inverse of a 2×2 matrix.

In multiplying a matrix by a number we imply that each entry in the matrix must be multiplied by that number. The value $ad - bc$ is known as the *determinant* of the matrix. For the matrix $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ the determinant is $4 \times 3 - 2 \times 1 = 10$.

3. Using the formula above, write down the determinant and inverse of the matrices. Check that the matrix multiplied by its inverse gives the identity matrix.

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$

Using matrices to solve equations

We are now able to solve the matrix equation on page 30, using the scheme set out on page 31.

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The determinant of the matrix $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ is $6 - 5 = 1$ and its inverse is then $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$.

Multiplying both sides of the matrix equation by the inverse matrix

gives $$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}.$$

From these equal matrices we obtain the solutions for x and y as $x = -4$ and $y = 7$.

EXERCISE SET 10

1. Solve the separate sets of simultaneous equations

(a) $2x + 7y = 3$, $x + 4y = 1$ (b) $3a - 2b = 14$, $a - b = 6$ (c) $3P + 2Q = 4$, $4P + 5Q = -11$

2. Use matrix methods to solve the equation $\mathbf{Ax} = \mathbf{h}$ where $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{h}$ takes in turn the values (a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ -8 \end{pmatrix}$ (c) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

3. Check that the inverse of the 3×3 matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

Put the equations $x + 2y + 3z = 3$, $2x + 4y + 5z = 4$, $3x + 5y + 6z = 5$ into matrix form and use the inverse matrix to solve the equations for x, y and z.

Matrices for geometry

Geometry is the study of shape, position and size. A figure which is moved to a new position, or has its shape or size altered, is said to undergo a *transformation*. Matrices are able to describe various transformations.

When a plane figure is moved to a new position without alteration of its shape or size, distances between points and angles between lines within the figure are preserved by this transformation. A transformation in which shape is retained and size altered preserves angles between lines within the figure but not distances between points. When size is retained the transformation preserves the area of the figure.

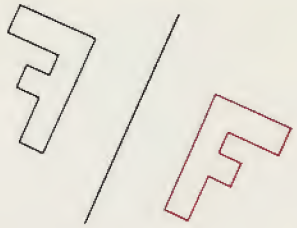
Transformations which preserve distances and angles

Mark a suitable figure—such as a figure **F** on paper and on a stiff sheet of cellophane make a trace of **F**. The trace **F** ensures that the transformations performed on the original **F** *do* preserve distance between points and angles between lines since the cellophane keeps the **F** rigid during the transformations.

What transformations of **F** are possible?

We begin each transformation with the **F** on the paper and the **F** on the cellophane coincident.

1. Mark a line on the paper and the cellophane. Flip the cellophane over and make the lines coincident. Draw the trace of **F** on the cellophane onto the paper with the original **F**.



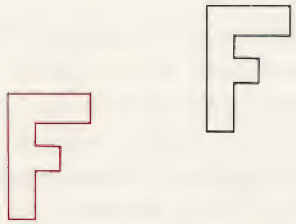
Such a transformation is known as a *reflection*.

2. With a pin through the cellophane and the paper, rotate the cellophane anticlockwise about the pin and then mark the **F** onto the paper.

× Pin

Such a transformation is known as a *rotation*.

3. The cellophane sheet is moved away without turning from **F** to a new position and the **F** marked onto the paper.

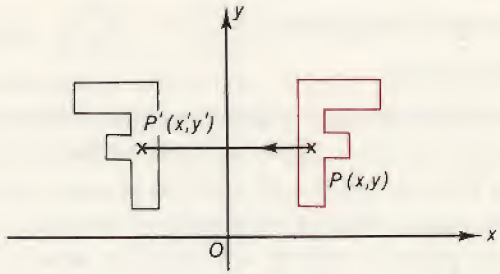


Such a transformation is known as a *translation*.

Transformations as matrices

In each transformation a point P on the original **F** becomes some point P' on the transformed **F** and we say that P is *mapped* into P' (writing $P \rightarrow P'$) by the transformation. We can consider these geometrical transformations as taking place in the co-ordinate plane.

1. *The reflection matrix*



When reflection takes place in the line Oy, any point $P(x, y)$ is mapped into a point $P'(x', y')$ where

$$\left.\begin{aligned} x' &= -x \\ y' &= y \end{aligned}\right\} \qquad (1)$$

since by the reflection the y coordinate is unchanged, but the x coordinate has the sign changed.

We may write the equations (1) in the form

$$\begin{aligned} x' &= -1x + 0y \\ y' &= 0x + 1y \end{aligned}$$

and now put these equations into matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (2)$$

The matrix form (2) describes the equations (1), and the matrix $\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ has the geometrical property of reflection in the line Oy.

For a particular point, say $(-2, 3)$, we can use the matrix to find the point into which it is mapped by the reflection. We use the

coordinates of the point $(-2, 3)$ as a column vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and

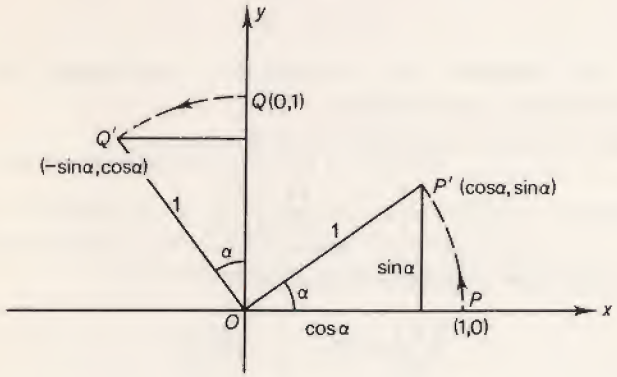
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

giving the mapped point as $(2, 3)$.

Find the maps of the points $(-1, -2)$; $(4, 3)$ by the matrix **R**. Show that the matrix which reflects in the line Ox is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and find the maps of the points $(-1, -2)$; $(4, 3)$.

2. *The rotation matrix*

Since we are considering a transformation which does not alter distances or angles within a figure we shall be able to determine the transformation if three points and their maps under the transformation are known; since, when a rigid figure is moved, the knowledge of how three points have moved will establish what transformation the figure has undergone. In the instance of a rotation about the origin O, O is mapped into itself so the origin and its map is known. For a rotation of α anticlockwise about O the point $P(1, 0) \rightarrow P'(\cos\alpha, \sin\alpha)$ and $Q(0, 1) \rightarrow Q'(-\sin\alpha, \cos\alpha)$.



We are looking for a matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which describes a transformation such that

$$(1, 0) \rightarrow (\cos\alpha, \sin\alpha)$$

and

$$(0, 1) \rightarrow (-\sin\alpha, \cos\alpha).$$

That is, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ and so $a = \cos\alpha$, $c = \sin\alpha$ by the equality of matrices.

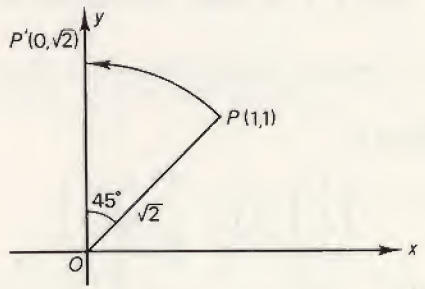
Also $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$ and therefore $b = -\sin\alpha$, $d = \cos\alpha$.

Thus the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ describes a rotation through an angle α anticlockwise about the origin. Generally we shall be considering anticlockwise rotations about the origin and this is to be understood unless stated otherwise.

If α is 45° the matrix would be $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

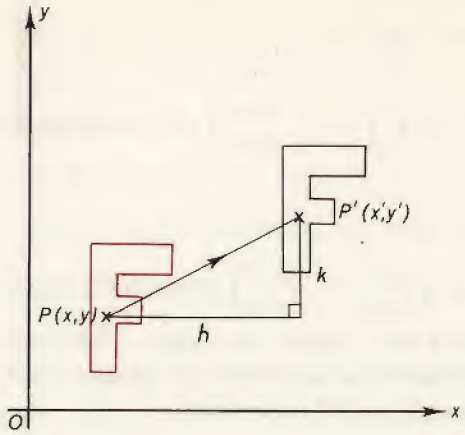
and this would take the point $(1, 1)$ to (x', y') where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}.$$



Find the matrices which rotate through 90° and 180° and find the respective maps of the points $(0, 2)$; $(-1, 1)$. By replacing α by $-\alpha$, find the matrix which will rotate through an angle α clockwise about O.

3. *The translation matrix*



A point $P(x, y)$ is translated horizontally through a distance h and vertically through a distance k to the point $P'(x', y')$. Then

$$x' = x + h$$
$$y' = y + k.$$

In matrix form these equations are

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

This time a 2×3 matrix is required as we have values h and k unattached to x and y. It is more convenient to have a square matrix, so the 2×3 matrix is made into a 3×3 as $\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ and the new matrix form is

$$\begin{pmatrix} x' \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

On multiplying out,

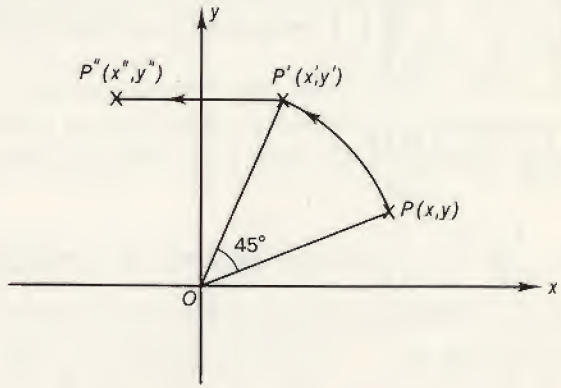
$$x' = x + h$$
$$y' = y + k$$
$$1 = 1,$$

the last equation fitting nicely!

Find the matrix which will translate a distance of 2 units horizontally and -1 unit vertically. What is the map of the point $(-2, 3)$?

Combining transformations

We have found matrices to describe certain transformations. As we can combine transformations by performing one after the other, can we use our matrices together to describe successive transformations?



What would be the matrix to rotate through 45° and then reflect in the line Oy? We know the matrix to rotate through 45° is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ so that from P to P' we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (1)$$

To reflect in the line Oy, the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; from P' to P'' then

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}. \qquad (2)$$

We wish to get from P to P'' and substitute for $\begin{pmatrix} x' \\ y' \end{pmatrix}$ from (1) into (2),

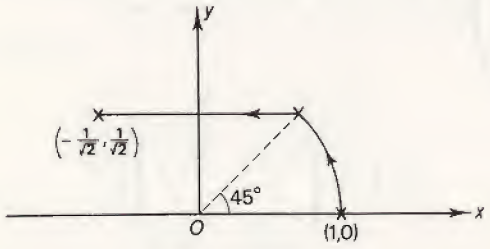
$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (3)$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ using matrix multiplication,}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ has the property of a rotation through 45° followed by a reflection in the line Oy. For the point (1, 0), the matrix gives the map as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$



By using matrix multiplication we can now find any combination of geometrical transformations. That matrix multiplication is

required in this situation arises from using matrix notation for the equations describing the transformations.

If we proceeded to use equations instead of matrices we would find that when one set of equations was used in another (as we have used matrices in going from equation (1) to (3)) the result is of the same form as we use in matrix multiplication (see example 5 of the next Exercise Set). Note that when we use substitution from (1) into (2) the order of transformations comes out backwards; that is, in (3), reading left to right as usual, we have the matrix for reflection $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, then the matrix for rotation $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and this represents a rotation through 45° *followed by* a reflection in the line Oy. If we had a transformation described by a matrix **A** *followed* by a transformation described by a matrix **B**, we need to compute **BA***.

EXERCISE SET 11

Use graph paper to check the mappings.

1. Find the matrix which will reflect in the line Oy followed by a rotation through 45°. What is the map of the point (1, 0)?

2. Find the matrix which will rotate through 90° followed by a further rotation of 180°. Show that this matrix is the matrix to rotate through 270°.

3. Find the matrix which will rotate through an angle α followed by a further rotation through an angle β. Such a matrix will be equivalent to a matrix to rotate through an angle of $\alpha + \beta$. State the matrix to rotate through $\alpha + \beta$ and hence establish the "Addition Formulae".

4. Find the matrix which will reflect in the line $y = x \tan \alpha$. (Consider using matrices which rotate through $-\alpha$ followed by a reflection in the

* In our initial situation of considering train (**T**) and bus (**B**) routes around a network we developed matrix multiplication from requiring a train journey followed by a bus journey and computing **TB**—*not* a backward order. This occurred because of our choice of the "from–to", reading (naturally) left to right of the matrix. A reverse of this choice would require a reverse order of matrices. Normally when matrices are combined under "row on column" matrix multiplication, they require this backwards order of writing.

line Ox followed by a rotation through $+\alpha$.) Find the map of the point $(1, 2)$ when it is reflected in the line $y = x \tan 30°$.

5. A point $P(x, y)$ is mapped into the point $P'(x', y')$ as described by the equations

$$x' = ax + by$$
$$y' = cx + dy.$$

Put these equations into matrix form.

The point $P'(x', y')$ is mapped into the point $P''(x'', y'')$ as described by the equations

$$x'' = px' + qy'$$
$$y'' = rx' + sy'.$$

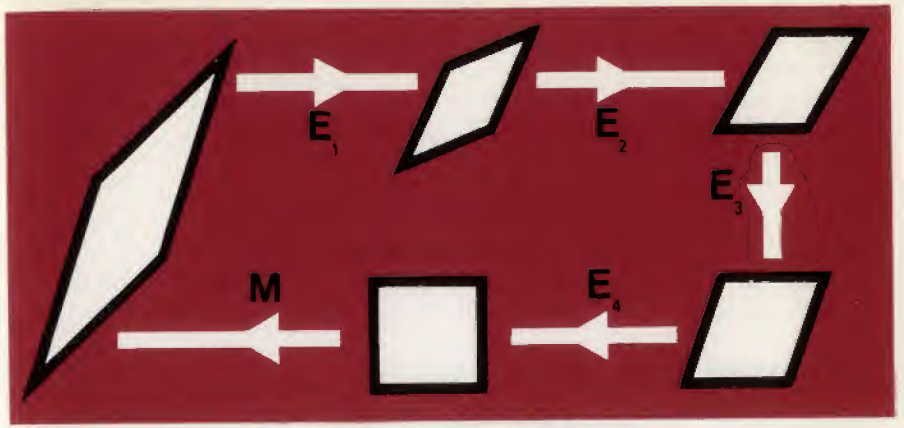
Put these equations into matrix form.

Find the equations describing the mapping of the point $P(x, y)$ into $P''(x'', y'')$ by substituting for x and y. Put these equations into matrix form.

Using the separate matrices and the resultant matrix establish the method of matrix multiplication.

6. To use the translation (3×3) matrix with another (2×2) matrix we shall need to enlarge the 2×2 matrix into a 3×3. The 2×2 reflection matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ becomes $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by placing 0 0 1 as a last row and column. What is the 3×3 matrix corresponding to the rotation through 45° matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$?

Find the matrix which will translate through 3 units horizontally and 1 unit vertically followed by a rotation through 45°, and find the map of the point $(1, 2)$.



More matrices for geometry

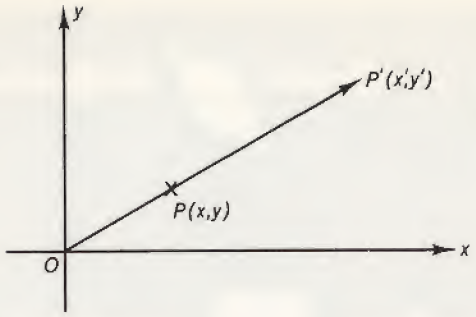
A transformation which preserves angles between lines

To investigate this type of transformation we can consider the figure **F** marked on paper and elastic strings fixed to some point and to points of **F**. Each of the elastic strings is stretched along its length by some determined factor, say k. The new **F** formed by the ends of the stretched elastic strings will have angles preserved and lengths altered proportionally, and will be similar to **F**.

Such a transformation is known as an *enlargement*.

On the coordinate plane, when we consider the origin as the point from which the enlargement takes place, a point $P(x, y)$ is mapped into the point $P'(x', y')$ where

$$x' = kx$$
$$y' = ky.$$



The matrix form of these equations is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

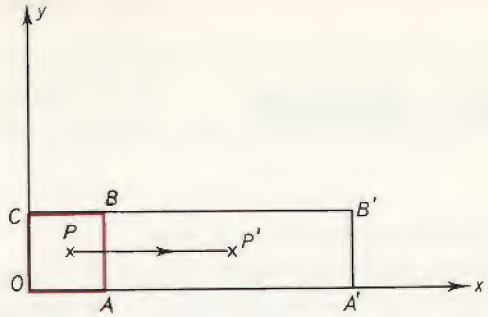
and the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ describes an enlargement of factor k. Find the matrices which will respectively enlarge by a factor of 2, reduce by a factor of $\frac{1}{2}$. Plot the unit square with vertices $O(0, 0)$; $A(1, 0)$; $B(1, 1)$ and $C(0, 1)$ and find the transformed square by using each of these matrices to map the vertex points.

A stretching transformation

If in the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ we let one of the k's be 1, we have the matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$. What is the geometrical property of the matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$?

We may discover the geometrical property of a matrix by considering how it transforms a particular figure. Taking the unit square with vertices at the points $O(0, 0)$; $A(1, 0)$; $B(1, 1)$ and $C(0, 1)$, we find the maps of these vertices by the matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. For the vertices used in turn with the matrix

$$\begin{matrix} & & O & A & B & C \end{matrix} \qquad \begin{matrix} O & A' & B' & C' \end{matrix}$$
$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & k & k & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$



The diagram suggests that the matrix has the geometrical property of stretching parallel to Ox. If $O\,A\,B\,C$ were an elastic sheet held at OC, then the matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ would pull AB to $A'B'$ where $\frac{OA'}{OA} = k$. By considering the point $P(x, y)$, then $P \rightarrow P'(x', y')$ by the matrix where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ y \end{pmatrix}$$

giving

$$\begin{aligned} x' &= kx \\ y' &= y. \end{aligned}$$

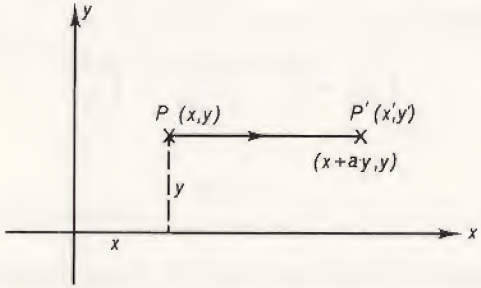
As the y coordinate is unaltered and the x coordinate multiplied by a factor of k, the matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ has the geometrical property of stretching parallel to Ox by a factor of k. This kind of transformation is known as a *linear stretch*. What is the geometrical property of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$?

Transformation which preserves area

When a figure is 'pushed over'—as a rectangle may be to a parallelogram—the transformation preserves area but not distances or angles.

Such a transformation is known as a *shear*.
On the coordinate plane any point $P(x, y)$ is moved horizontally through a distance which is proportional to its ordinate value; that is, the further the point is away from Ox the further it is moved across. If the factor of proportionality is a, then the point $P(x, y)$ becomes $P'(x', y')$ where

$$x' = x + ay$$
$$y' = y.$$



The matrix form of these equations is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and the matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ describes a shear of factor a taken parallel to Ox.

Find the matrices which will shear respectively by a factor of 2 and -2 parallel to Ox. Plot the unit square and its transform for each matrix.

Find the matrices which will shear respectively by a factor of 3 and -3 parallel to Ox. Plot the points $A(1, 1)$; $B(3, 1)$; $C(3, 2)$ and $D(1, 2)$ and find the transformed figure.

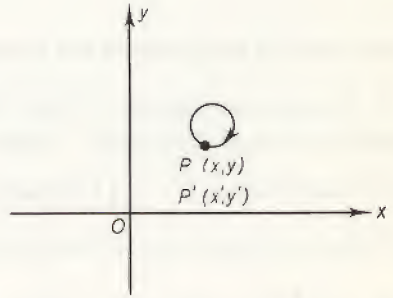
What would be the matrix to shear parallel to Oy by a factor of b?

The unchanging transformation

A transformation in which nothing is changed is known as the *identity* transformation.

On the coordinate plane a point $P(x, y)$ stays as (x, y), so $P'(x', y')$ is given by

$$x' = x$$
$$y' = y.$$



The matrix form of these equations is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix and has the property of making no change.

Reversing a transformation

If a figure has undergone a transformation by the matrix **A**, and another transformation by the matrix **B** will return the figure to its original state, then the transformation **B** is known as the *inverse* of **A**.

The matrix which reflects in Oy is $\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and to restore points to their original state we would reflect in Oy again. Hence the inverse matrix of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

When a matrix and its inverse are taken together, the result is no change—represented by the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence a reflection in Oy followed by a reflection in Oy must give no change, and

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

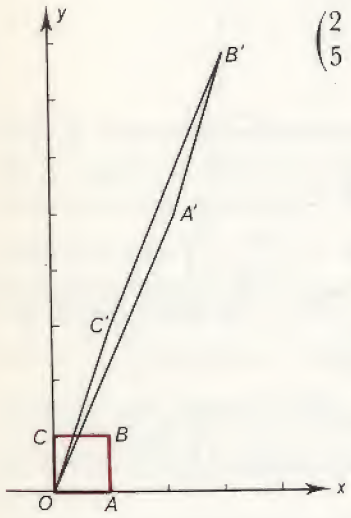
The inverse of a shear matrix $\begin{pmatrix}1 & a\\0 & 1\end{pmatrix}$ is $\begin{pmatrix}1 & -a\\0 & 1\end{pmatrix}$ since $\begin{pmatrix}1 & -a\\0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. What would be the inverse of the linear stretch matrix $\begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$ and of the shear matrix $\begin{pmatrix}1 & 0\\b & 1\end{pmatrix}$?

The geometrical property of the inverse

In the cases of inverse of the reflection and shear matrices, the geometry is straightforward. What of the geometry of the inverse of the matrix $\mathbf{A} = \begin{pmatrix}2 & 1\\5 & 3\end{pmatrix}$? We know the inverse of $\mathbf{A}$ is $\begin{pmatrix}3 & -1\\-5 & 2\end{pmatrix}$ and this was found by a succession of $\mathbf{E}$ matrices, which were (p. 32):

$$\mathbf{E}_1 = \begin{pmatrix}\frac{1}{2} & 0\\0 & 1\end{pmatrix};\quad \mathbf{E}_2 = \begin{pmatrix}1 & 0\\-5 & 1\end{pmatrix};\quad \mathbf{E}_3 = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix};\quad \mathbf{E}_4 = \begin{pmatrix}1 & -\frac{1}{2}\\0 & 1\end{pmatrix}$$

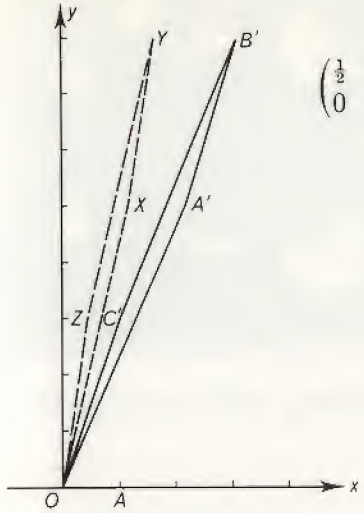
What transformation takes place if we use $\mathbf{A}$ on the unit square $O\ A\ B\ C$?



$$\begin{matrix} & O & A & B & C \end{matrix} \qquad \begin{matrix} & O' & A' & B' & C' \end{matrix}$$
$$\begin{pmatrix}2 & 1\\5 & 3\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\end{pmatrix} \rightarrow \begin{pmatrix}0 & 2 & 3 & 1\\0 & 5 & 8 & 3\end{pmatrix}$$

The inverse matrix of $\mathbf{A}$ would restore $O\ A'\ B'\ C'$ to $O\ A\ B\ C$. We now see how the $\mathbf{E}$ matrices find the inverse matrix by systematically returning $O\ A'\ B'\ C'$ to $O\ A\ B\ C$.

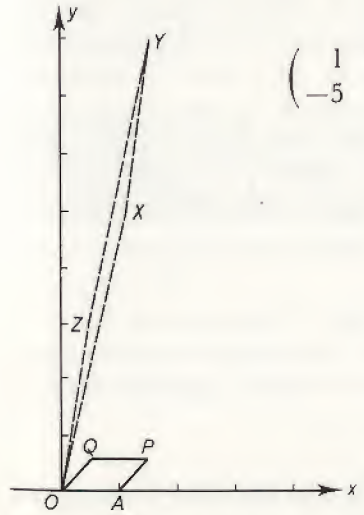
$\mathbf{E}_1 = \begin{pmatrix}\frac{1}{2} & 0\\0 & 1\end{pmatrix}$ and transforms $O\ A'\ B'\ C'$ as



$$\begin{matrix} & O & A' & B' & C' \end{matrix} \qquad \begin{matrix} & O & X & Y & Z \end{matrix}$$
$$\begin{pmatrix}\frac{1}{2} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 2 & 3 & 1\\0 & 5 & 8 & 3\end{pmatrix} \rightarrow \begin{pmatrix}0 & 1 & \frac{3}{2} & \frac{1}{2}\\0 & 5 & 8 & 3\end{pmatrix}.$$

$\mathbf{E}_1$ is a linear stretch matrix which since $k = \frac{1}{2}$, pushes $O\ A'\ B'\ C'$ toward Oy parrallel to Ox by a factor of $\frac{1}{2}$ making the x coordinate of A' be 1 (over A).

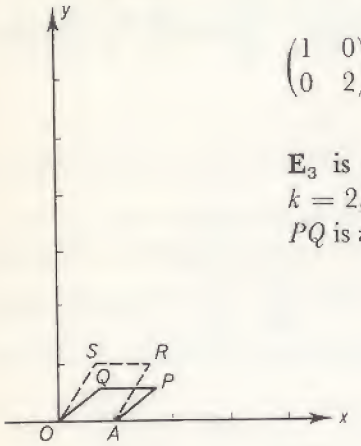
$\mathbf{E}_2 = \begin{pmatrix}1 & 0\\-5 & 1\end{pmatrix}$ and transforms $O\ X\ Y\ Z$ as



$$\begin{matrix} & O & X & Y & Z \end{matrix} \qquad \begin{matrix} & O & A & P & Q \end{matrix}$$
$$\begin{pmatrix}1 & 0\\-5 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & \frac{3}{2} & \frac{1}{2}\\0 & 5 & 8 & 3\end{pmatrix} \rightarrow \begin{pmatrix}0 & 1 & \frac{3}{2} & \frac{1}{2}\\0 & 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

$\mathbf{E}_2$ is the inverse of a shear matrix which shears $O\ X\ Y\ Z$ toward Ox to make OX be along OA.

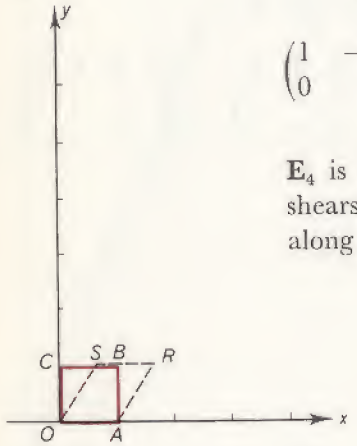
$\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and transforms $O\ A\ P\ Q$ as



$$\begin{matrix} & O & A & P & Q \\ \end{matrix} \qquad \begin{matrix} & O & A & R & S \\ \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$\mathbf{E}_3$ is a linear stretch matrix which, since $k = 2$, stretches $O\ A\ P\ Q$ parallel to Oy until PQ is along $y = 1$, the top of the unit square.

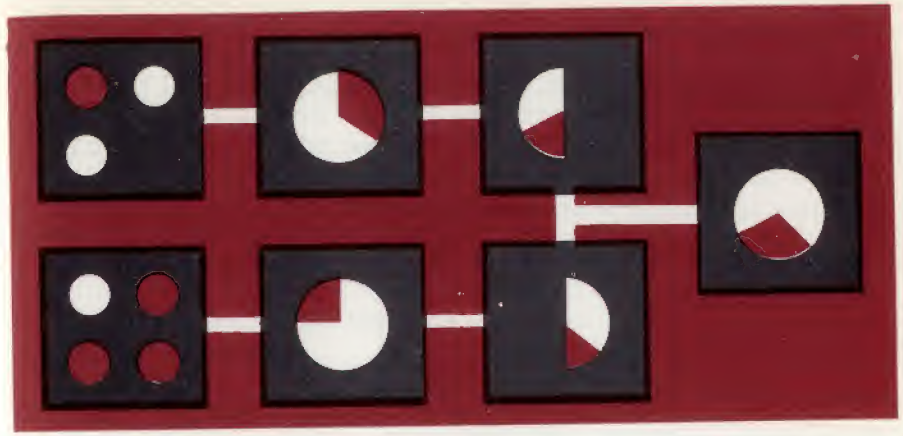
$\mathbf{E}_4 = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$ and transforms $O\ A\ R\ S$ as



$$\begin{matrix} & O & A & R & S \\ \end{matrix} \qquad \begin{matrix} & O & A & B & C \\ \end{matrix}$$
$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$\mathbf{E}_4$ is the inverse of a shear matrix which shears $O\ A\ R\ S$ toward Oy to make OS lie along OC, giving the unit square again.

Find the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ and illustrate by diagrams how the $\mathbf{E}$ matrices restore the mapping of the unit square by this matrix to the unit square.



Matrices for probability

A black bag X has 3 marbles: 2 red and 1 white. A black bag Y contains 4 marbles: 3 red and 1 white. If the bags are indistinguishable and held out so that I choose a bag and then a marble in it, what is the probability* I pick out a red marble?

When the two bags are held out there is an equal probability of choosing X or Y; that is, the probability of choosing X is a $\frac{1}{2}$ and the probability of choosing Y is a $\frac{1}{2}$. Having chosen bag X, there is a probability of $\frac{2}{3}$ I pick out a red marble (since there are 2 red marbles in 3 in bag X). As the two events, choosing bag X and then choosing a marble out of the bag, are independent,* the probability of picking a red marble from bag X is $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. For bag Y there is a probability of $\frac{3}{4}$ (3 red marbles in 4 in bag Y) of picking out a red marble. As choosing bag Y and then any marble in it are independent events, the probability of picking a red marble from bag Y is $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.

The two sequences—choosing bag X or Y and then a marble—are exclusive* and hence the probability of choosing a red marble when the bags are held out is

$$\tfrac{1}{2} \times \tfrac{2}{3} + \tfrac{1}{2} \times \tfrac{3}{4} = \tfrac{1}{3} + \tfrac{3}{8} = \tfrac{17}{24}.$$

* For an explanation of these terms and their use in mathematics see *Probability and Chance*, pp. 7–9, 32–41, Number 15 of this series.

The probability vector for choosing a bag is $\mathbf{B} = \begin{matrix} X \\ Y \end{matrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

The probability matrix describing the contents of the bags is

$$\mathbf{M} = \begin{matrix} \\ R \\ W \end{matrix}\begin{matrix} X & Y \\ \end{matrix}\begin{pmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}.$$

We can find the probability of choosing a red marble by using these matrices as

$$\begin{matrix} \\ R \\ W \end{matrix}\begin{matrix} X & Y \\ \end{matrix}\begin{pmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}\begin{matrix} X \\ Y \end{matrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{matrix} R \\ W \end{matrix}\begin{pmatrix} \frac{2}{3} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2} \\ \frac{1}{3} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} \end{pmatrix} = \begin{matrix} R \\ W \end{matrix}\begin{pmatrix} \frac{17}{24} \\ \frac{7}{24} \end{pmatrix}.$$

Again our rule for matrix multiplication is the method we can use when finding probabilities of events which are independent and exclusive. Note here we have used the backward order **MB**—choosing a bag followed by a marble in the bag.

EXERCISE SET 12

1. In a black bag X there are 2 red and 3 white marbles. In the black bag Y there are 3 red and 1 white marble. If the two bags are indistinguishable and held out, what is the probability a red marble is chosen?

2. A cube A has two faces marked with a 1, two faces marked with a 3, one face marked 4 and one face marked 5. A cube B has one face marked 1, one face marked 2, three faces marked 3 and one face marked 5. If the cubes are put in a black bag and then one cube is picked out and then rolled, what is the probability of obtaining a 1, a 2, a 3, a 4, or a 5?

3. Each morning I walk to the bus stop up or down the road. The probability is $\frac{1}{3}$ I go up the hill to the bus stop and at this stop buses 15 or 15A take me into town and there is an equal probability of catching either. The downhill stop is only for bus 15. What is the probability I take bus 15 into town?

4. In one black bag there are 2 red, 1 white and 2 black marbles, in another black bag there are 1 red, 3 white and 1 black marble and in a third black bag, 2 red and 3 black marbles. If the bags are held out and one bag chosen and then a marble is picked out of the bag, what is the probability it is a black marble?

Probability chains

A maths and physics student begins the term by working at his maths. He finds though that he has more physics than maths to do and if on any day he works at maths, he will be on physics the next day. Whereas if he is working at physics on any day there is an even chance he will be working at maths or physics the next day.

The starting vector is $\begin{matrix} M \\ P \end{matrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as he starts the term working at maths. The day-to-day matrix is

$$\begin{matrix} \\ \text{tomorrow} \\ \\ \end{matrix}\begin{matrix} \mathbf{M} & M & P \\ \end{matrix}\quad\text{today}$$
$$\begin{matrix} M \\ P \end{matrix}\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$$

where the first column indicates that if the student works at maths today he will certainly be doing physics tomorrow, and the second column indicates that if he is working at physics today there are probabilities of $\frac{1}{2}$ he will be working at maths or physics tomorrow.

For the first day of term the probabilities of working at maths or physics are given by $\begin{matrix} M \\ P \end{matrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For the second day of term the probabilities are given by matrices as

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{matrix} M \\ P \end{matrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

showing he has definitely changed to physics!

For the third day we use these probabilities and the matrix **M** again as

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

For the fourth day the probabilities are

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

The fifth day probabilities are

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{5}{8} \end{pmatrix}$$

and for the sixth day

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{3}{8}\\ \frac{5}{8}\end{pmatrix} = \begin{pmatrix}\frac{5}{16}\\ \frac{11}{16}\end{pmatrix}.$$

What would be the probability of the student working at maths on the last day of term? If the term were 101 days we need to continue this process 101 times! But, what is happening is that we begin with the probability vector $\begin{pmatrix}1\\ 0\end{pmatrix}$ and continually multiply by the matrix **M**. For the fourth day the probabilities are given by

$$\mathbf{M}^3\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

and so on the 101st day, probabilities are given by

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}^{100}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

Can we easily find the value of $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}^{100}\begin{pmatrix}1\\ 1\end{pmatrix}$? The results we have of the successive probabilities are a pointer. They are:

	Maths	Physics		Maths	Physics
1st day	1	0		1	0
2nd day	0	1		0	1
3rd day	$\frac{1}{2}$	$\frac{1}{2}$		0·5	0·5
4th day	$\frac{1}{4}$	$\frac{3}{4}$	or in decimal form	0·25	0·75
5th day	$\frac{3}{8}$	$\frac{5}{8}$		0·375	0·625
6th day	$\frac{5}{16}$	$\frac{11}{16}$		0·3125	0·6875
7th day	$\frac{11}{32}$	$\frac{21}{32}$		0·344	0·636

In each column the values alternate in size—larger then smaller, and also the next larger value is not so large as its predecessor nor the next smaller value so small as its predecessor. So on the eighth day we would expect a probability of maths smaller than 0·344 but larger than 0·3125 and for physics a probability larger than 0·636 and smaller than 0·6875.

The eighth day probabilities are:

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0{\cdot}344\\ 0{\cdot}636\end{pmatrix} = \begin{pmatrix}0{\cdot}318\\ 0{\cdot}682\end{pmatrix}$$

justifying our expectations!

Should this fluctuation in values continue it implies that eventually there will be probabilities for maths and physics which the matrix $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}$ will not change.

Looking at the results, we may guess these probabilities to be $\frac{1}{3}(\doteq 0{\cdot}33)$ for maths and $\frac{2}{3}(\doteq 0{\cdot}67)$ for physics. If our guess is correct, then the matrix $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}$ taken with these values should again give the same values; and it does!

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}.$$

Here we have a situation in which a matrix describes the probabilities of change from any day to the next day. Such a matrix is known as a *Transition Matrix*. The outcome of the situation is found by multiplying the matrix with itself as a chain. It may occur that there will be values such that when the matrix is taken with these values they are repeated.

In the case of the maths and physics student, the transition matrix $\mathbf{M} = \begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}$, describing the day-to-day probabilities of the student working at maths or physics, has the property of not changing the probability vector $\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}\begin{matrix}M\\ P\end{matrix}$. It is true that these probabilities will not be reached by the 101st day of term—but the difference will not be great! There is then a probability of $\frac{1}{3}$ that the student will be studying maths on the last day of term—particularly if the term is endless!

The column vector $\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}$ which has the special property of being unchanged when multiplied by the matrix, is known as an *eigenvector* of the matrix. It may be that the eigenvector when multiplied by the matrix is given again but multiplied by a number, such as in the case

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix}.$$

This time an eigenvector is $\begin{pmatrix}1\\ -1\end{pmatrix}$ and the value $-\frac{1}{2}$, the multiply-

ing number, is known as the *eigenvalue* (or *latent root*). In the instance of the eigenvector $\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}$, the eigenvalue is 1. When a situation can be represented in matrix form, the eigenvector of the matrix gives the invariant, that is unchanging, values of the situation. If the eigenvalue is 1, the eigenvector represents the 'steady-state' position.

In the situation of the maths and physics student the existence of an eigenvector was discovered by looking at the values obtained. There is a direct method of finding eigenvectors of a matrix.

If $\begin{pmatrix}x\\ y\end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}$ then

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \lambda\begin{pmatrix}x\\ y\end{pmatrix}$$

since the matrix taken with an eigenvector gives the eigenvector again, but the eigenvector could be multiplied by a number, λ, the eigenvalue.

From the matrix form, multiplying out

$$\begin{pmatrix}\frac{1}{2}y\\ x + \frac{1}{2}y\end{pmatrix} = \begin{pmatrix}\lambda x\\ \lambda y\end{pmatrix}.$$

So $$\frac{1}{2}y = \lambda x$$

and $$x + \frac{1}{2}y = \lambda y.$$

Rearranging these equations as

$$-\lambda x + \tfrac{1}{2}y = 0 \qquad (1)$$
$$x + y(\tfrac{1}{2} - \lambda) = 0 \qquad (2)$$

we see that $x = 0$, $y = 0$ are solutions. But these values are not informative; any 2×2 matrix taken with $\begin{pmatrix}0\\ 0\end{pmatrix}$ gives $\begin{pmatrix}0\\ 0\end{pmatrix}$. The graphs of $-\lambda x + \frac{1}{2}y = 0$ and $x + y(\frac{1}{2} - \lambda) = 0$ are, for any value of λ, two straight lines meeting at the origin. They cannot meet at any other point unless they coincide. If an eigenvector exists then the two equations must be equivalent, so the gradient of the first must equal that of the second. Then

$$\frac{\lambda}{\frac{1}{2}} = -\frac{1}{\frac{1}{2} - \lambda}.$$

Hence $$\lambda(\tfrac{1}{2} - \lambda) = -\tfrac{1}{2},$$
$$\lambda - 2\lambda^2 = -1$$

or $$2\lambda^2 - \lambda - 1 = 0.$$

This equation is known as the *characteristic* equation of the matrix.

Then $(\lambda - 1)(2\lambda + 1) = 0$ and $\lambda = 1$ or $-\frac{1}{2}$, the two eigenvalues we found in the example.

Putting $\lambda = 1$ in (1) gives $-x + \frac{1}{2}y = 0$ and in (2) gives $x - \frac{1}{2}y = 0$; the same equation as $-x + \frac{1}{2}y = 0$. There are thus many eigenvectors $\begin{pmatrix}x\\ y\end{pmatrix}$ of the matrix $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}$, given when x and y satisfy $-x + \frac{1}{2}y = 0$; for example $x = 1, y = 2$ and $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}1\\ 2\end{pmatrix}$.

In general we can give the eigenvector corresponding to the eigenvalue $\lambda = 1$ as $\begin{pmatrix}t\\ 2t\end{pmatrix}$ since $x = t, y = 2t$ satisfies $-x + \frac{1}{2}y = 0$ for all values of t. But in the situation of the maths and physics student we are looking for the *probability* eigenvector $\begin{pmatrix}x\\ y\end{pmatrix}$ and in this instance $x + y = 1$. With $x = t$, $y = 2t$ and $x + y = 1$ we have $3t = 1$, $t = \frac{1}{3}$ and $x = \frac{1}{3}, y = \frac{2}{3}$, giving the eigenvector as $\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}$.

When $\lambda = -\frac{1}{2}$, equation (1) gives $\frac{1}{2}x + \frac{1}{2}y = 0$ or $x + y = 0$ and equation (2) also is $x + y = 0$. Any values of x and y such that $x + y = 0$ will give further eigenvectors of the matrix $\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}$.

For example $x = 2, y = -2$ and

$$\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}2\\ -2\end{pmatrix} = \begin{pmatrix}-1\\ 1\end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix}2\\ -2\end{pmatrix}.$$

Note how the eigenvalue $-\frac{1}{2}$, is taken out as a factor before the eigenvector $\begin{pmatrix}2\\ -2\end{pmatrix}$ is repeated. The general eigenvector corresponding to the eigenvalue $\lambda = -\frac{1}{2}$ is $\begin{pmatrix}t\\ -t\end{pmatrix}$. Again we require the eigenvector to be a probability vector with $x + y = 1$, but with $x + y = 0$ there are no solutions for x and y in this case.

EXERCISE SET 13

1. Find the eigenvalues and eigenvectors of the matrices

(a) $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

2. A student has study habits such that if one day he does study, there is a probability of $\frac{2}{3}$ he will not study the next day. If he does no study one day there is a probability of $\frac{1}{2}$ he will study the next day. He begins the term by studying.

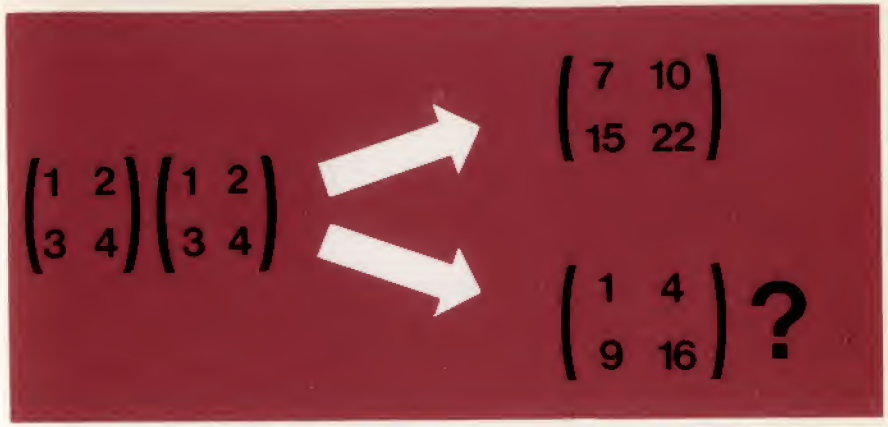
(a) If he studies one day what is the probability he studies the next day?

(b) Set up the transition matrix describing the study habits of the student.

(c) What are the probabilities the student is studying on the second, third and fourth days of term?

(d) If the term is considered endless, what is the probability the student studies on the last day of term?

3. It is observed that a teacher is never late for his first lesson of term. However, if he is late for one lesson there is a probability of $\frac{3}{4}$ he will be on time for the next. Whereas if he is on time for the lesson there is a probability of $\frac{1}{2}$ he will be on time for the next. What is the probability the teacher is late for his last lesson of term?



The use of matrices

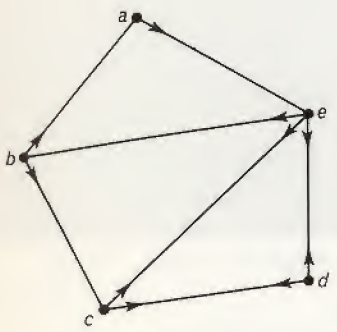
A look back

A matrix is a table of information; it is a compact store of data. Many situations make use of matrices since they present information in a clear and easily readable form. In some situations a matrix is combined with another matrix and there are two particular methods of combination. One is where corresponding entries in the matrices are added, this method being called addition of matrices, and one where corresponding entries in the row of one matrix and the column of the next matrix are multiplied and then added, this method being called multiplication of matrices. There are several instances where multiplication of matrices is used—in networks, for items of information, for costing, as geometrical transformations, in solving linear equations and in probability.

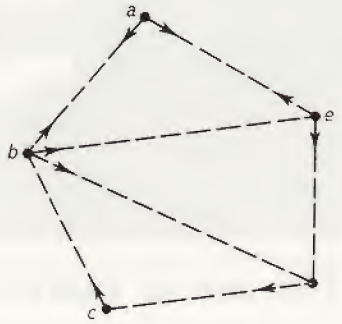
A look forward

A situation described by a matrix has been presented in a concise form. It is possible to sift information and also gain more information about the situation if it is possible to combine matrices in some way. The instances we have looked at have used the method we have

called multiplication, but this need not be the only method. For example, consider the train and bus routes between five towns a, b, c, d and e as described by the networks:



Train routes



Bus routes

The matrix, **T**, describing the train routes is

$$\mathbf{T} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{l} \text{to} \\ \\ \\ \\ \\ \text{from} \end{array}.$$

The matrix, **B**, describing the bus routes is

$$\mathbf{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{l} \text{to} \\ \\ \\ \\ \\ \text{from} \end{array}.$$

Let us find the result of **TB** by simply multiplying corresponding *entries* of the two matrices—a natural way, similar to the addition of two matrices by adding corresponding entries.

Then

$$\mathbf{TB} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

What information does the resultant matrix give? Between the five towns it states the routes

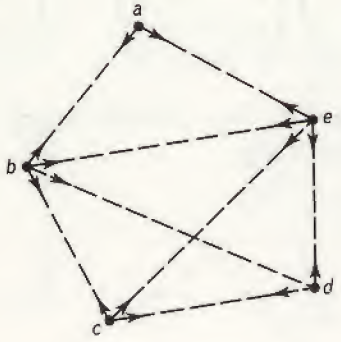
and these are routes between the towns for which a train *or* a bus may be taken.

Strictly we should put some symbol between the matrices of **T** and **B** to which we identify this mode of combination. Let the symbol be $\otimes$ and then $\mathbf{T} \otimes \mathbf{B}$ implies we combine **T** and **B** by multiplying corresponding entries.

This time let us combine **T** and **B** by $\oplus$, which means add corresponding entries but according to the rules $0 + 0 = 0$, $1 + 0 = 1$, $0 + 1 = 1$ and $1 + 1 = 1$. Then

$$\mathbf{T} \oplus \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

What information does the resultant matrix give? Between the five towns it states the routes

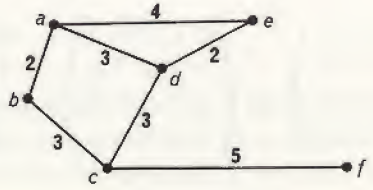


and these are routes between the towns where there exists a train *or* bus *or* both connections.

As a third illustration of different ways of combining two matrices, consider $\mathbf{T} \star \mathbf{B}$ where we add corresponding entries of $\mathbf{T}$ and $\mathbf{B}$ but according to the rules $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$ and $1 + 1 = 0$. What is the resultant matrix in this case and what information about the train and bus routes between the five towns does it describe?

Although the most widely used and applicable way of combining two matrices is the row on column method used in the various sections of this book, other means are possible. This versatility of method is a source of exploration.

Six towns *a*, *b*, *c*, *d*, *e* and *f* are connected by roads with mileage as given by the network



The matrix $\mathbf{M}$ describing these connections with mileage between towns is

$$\mathbf{M} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} - & 2 & - & 3 & 4 & - \\ 2 & - & 3 & - & - & - \\ - & 3 & - & 3 & - & 5 \\ 3 & - & 3 & - & 2 & - \\ 4 & - & - & 2 & - & - \\ - & - & 5 & - & - & - \end{pmatrix} \end{array}$$

where "–" indicates the towns are not connected.

We are to combine $\mathbf{M}$ with itself under a method we will call $\odot$, such that we use our row on column way of taking entries together but add (rather than multiply) whenever two numbers correspond, putting a "–" otherwise, and then choosing the smallest of the separate additions; that is

$$\mathbf{M} \odot \mathbf{M} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} - & 2 & - & 3 & 4 & - \\ 2 & - & 3 & - & - & - \\ - & 3 & - & 3 & - & 5 \\ 3 & - & 3 & - & 2 & - \\ 4 & - & - & 2 & - & - \\ - & - & 5 & - & - & - \end{pmatrix} \end{array} \odot \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} - & 2 & - & 3 & 4 & - \\ 2 & - & 3 & - & - & - \\ - & 3 & - & 3 & - & 5 \\ 3 & - & 3 & - & 2 & - \\ 4 & - & - & 2 & - & - \\ - & - & 5 & - & - & - \end{pmatrix} \end{array}$$

and from row $a(-\ 2\ -\ 3\ 4\ -)$ with column $a \begin{pmatrix} - \\ 2 \\ - \\ 3 \\ 4 \\ - \end{pmatrix}$ we would work as $- + -, 2 + 2, - + -, 3 + 3, 4 + 4, - + - = -, 4, -, 6, 8, -$ and choose 4 (as the smallest value) as the result.

For row *a*, column *b*

$$(-\ 2\ -\ 3\ 4\ -) \begin{pmatrix} 2 \\ - \\ 3 \\ - \\ - \\ - \end{pmatrix} = - + 2, 2 + -, - + 3, 3 + -, 4 + -, - + -$$

$$= -, -, -, -, - \text{ giving } -.$$

For row a, column c the working is

$$(- \ 2 \ - \ 3 \ 4 \ -)\begin{pmatrix} - \\ 3 \\ - \\ 3 \\ - \\ 5 \end{pmatrix} = - + -, 2 + 3, - + -, 3 + 3, 4 + -, - + 5$$

$$= -, 5, -, 6, -, - \text{ giving } 5.$$

By continuing

$$\mathbf{M} \odot \mathbf{M} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 4 & - & 5 & 6 & 5 & - \\ - & 4 & - & 5 & 6 & 8 \\ 5 & - & 6 & - & 5 & - \\ 6 & 5 & - & 4 & 7 & 8 \\ 5 & 6 & 5 & 7 & 4 & - \\ - & 8 & - & 8 & - & 10 \end{pmatrix} \end{array}.$$

The matrix $\mathbf{M} \odot \mathbf{M}$ describes the shortest distance between two towns via another town, the "–" entries implying no connection possible.

The power and usefulness of matrices lies in their ability to organise and present masses of data, to extract further information by methods of combining the matrices and in the use of the eigenvector.

Matrix methods deal with problems we can already solve by other means. We use a nutcracker to crack a nut but in matrices we have a sledgehammer and sometimes the nut can be hard.

For further reading

COULSON, A. E., *An Introduction to Matrices*, Longmans 1965.

GIBSON, G. R. and MAYATT, J., *First Stages in Matrices*, U.L.P., 1965

BRANFIELD, J. R. and BELL, A. W., *Matrices and their Applications*, Macmillan 1970.

BRAND, T. and SHERLOCK, A., *Matrices: Pure and Applied*, Arnold 1970.

Solutions to the exercises

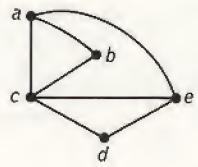
page 3 Brighton to Oxford 97 miles.
Manchester to London 184 miles.

EXERCISE SET 1

(a)
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

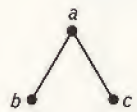
(b)
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

(c)



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

2. (a)



(b)

(c)

3. There are 3 roads at c.

4. If a is connected to b, then b is connected to a, etc.

EXERCISE SET 2

1. From a it may be possible to get to b, but now not from b to a.
Sum of row entries give number of routes *from* the town.
Sum of column entries give number of routes *into* the town.

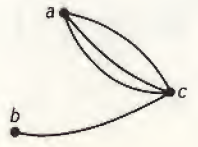
2. (a)
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

(b)
$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} b & c \end{array} \\ \begin{pmatrix} 3 & 2 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \end{array}$$

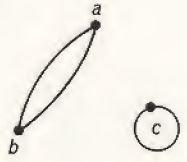
(c)
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

(d)
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

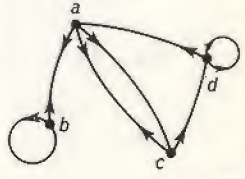
3. (a)



(b)



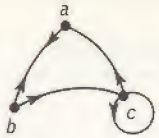
(c)



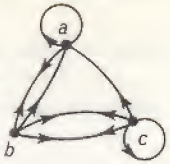
EXERCISE SET 3

1. $\begin{pmatrix}1&1&0\\2&0&2\\1&2&1\end{pmatrix}$

with



gives



2. The matrix to describe 4 towns, each to have a route to every other town is

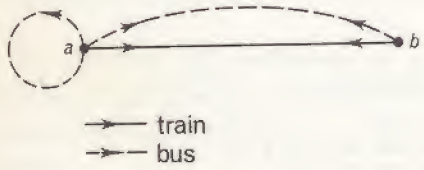
$$\begin{array}{c} \\ a\\ b\\ c\\ d\end{array}\begin{array}{c}\begin{array}{cccc}a&b&c&d\end{array}\\\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}\end{array}$$

Then $\begin{pmatrix}0&1&0&1\\0&0&1&0\\0&0&0&1\\0&0&1&0\end{pmatrix} + \mathbf{M} = \begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$

$\mathbf{M} = \begin{pmatrix}0&0&1&0\\1&0&0&1\\1&1&0&0\\1&1&0&0\end{pmatrix}$ giving the extra routes required.

EXERCISE SET 4

1. (a)



(b) $\mathbf{BT} = \begin{pmatrix}1&1\\0&1\end{pmatrix}$ Describes journeys possible first by bus and then train.

(c) $\mathbf{T}^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ Towns between which a train ride followed by a train ride is possible.

(d) $\mathbf{TBT} = \begin{pmatrix}0&1\\1&1\end{pmatrix}$ Describes possible journeys between towns of a train ride, followed by bus, followed by train.

2. (a) $\begin{pmatrix}1&2\\0&2\end{pmatrix}$ (b) $\begin{pmatrix}2&0\\3&1\end{pmatrix}$ (c) $\begin{pmatrix}3&2&3\\1&1&1\\2&0&5\end{pmatrix}$ (d) $\begin{pmatrix}6&3&4\\3&9&5\\3&1&2\end{pmatrix}$

(e) $\begin{pmatrix}2&1\\0&1\end{pmatrix}\begin{pmatrix}2&2\\2&1\end{pmatrix} = \begin{pmatrix}6&5\\2&1\end{pmatrix}$ or $\begin{pmatrix}3&1\\1&1\end{pmatrix} + \begin{pmatrix}3&4\\1&0\end{pmatrix} = \begin{pmatrix}6&5\\2&1\end{pmatrix}$

3. (a) $\mathbf{B} = \begin{array}{c}\\a\\b\\c\end{array}\begin{array}{c}\begin{array}{ccc}a&b&c\end{array}\\\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}\end{array}$ $\mathbf{W} = \begin{array}{c}\\a\\b\\c\end{array}\begin{array}{c}\begin{array}{ccc}a&b&c\end{array}\\\begin{pmatrix}0&0&1\\0&1&1\\1&0&0\end{pmatrix}\end{array}$

(b) $\mathbf{BW} = \begin{pmatrix}1&1&1\\0&0&1\\0&1&1\end{pmatrix}$

(c) $\mathbf{WB} = \begin{pmatrix}0&1&0\\1&1&0\\0&1&1\end{pmatrix}$

(d) $\mathbf{W}^2 = \begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}$

(e) $\mathbf{WBW} = \begin{pmatrix}0&1&1\\0&1&2\\1&1&1\end{pmatrix}$

(f) $\mathbf{B}^3 = \begin{pmatrix}1&1&1\\1&1&0\\0&1&1\end{pmatrix}$

$$\begin{array}{c}\\\text{Andrew}\\\text{Brenda}\end{array}\begin{array}{c}\begin{array}{cccc}\text{doughnut}&\text{chocolate bar}&\text{milk}&\text{price}\end{array}\\\begin{pmatrix}1&0&1\\0&2&2\end{pmatrix}\end{array}\begin{pmatrix}2\\3\\5\end{pmatrix} = \begin{pmatrix}7\\16\end{pmatrix}$$

EXERCISE SET 5

(a) $\begin{pmatrix}7&3\\12&9\end{pmatrix}$ (b) $\begin{pmatrix}18\\8\end{pmatrix}$ (c) not possible (d) $\begin{pmatrix}5&2&11\\5&6&13\\4&0&8\end{pmatrix}$

(e) 14 (f) $\begin{pmatrix}1&2&3\\2&4&6\\3&6&9\end{pmatrix}$ (g) not possible (h) $\begin{pmatrix}4&2&0\\10&8&4\\24&18&8\end{pmatrix}\begin{pmatrix}10\\20\\14\end{pmatrix} = \begin{pmatrix}80\\316\\712\end{pmatrix}$

Two matrices may be multiplied only when the number of *columns* in the first matrix equals the number of *rows* in the second matrix. If the first matrix is an $\mathbf{m} \times \mathbf{n}$ matrix, then for multiplication to be possible the second matrix must be of the form $\mathbf{n} \times \mathbf{p}$. The resultant matrix is then $\mathbf{m} \times \mathbf{p}$.

EXERCISE SET 6

$$\begin{pmatrix}2&4&4&12&6&2&0\\4&0&0&12&12&0&2\\0&2&2&0&0&2&2\end{pmatrix}\begin{array}{l}\text{antirrhinums}\\\text{stocks}\\\text{sweet-williams}\\\text{chrysanthemums}\\\text{polyanthuses}\\\text{mesembryanthemums}\\\text{asters}\end{array}\begin{array}{c}\text{price}\\\begin{pmatrix}12\\10\\12\\5\\3\\10\\12\end{pmatrix}\end{array}$$

(antirrhinums: ($\frac{1}{2}$ doz.), stocks: ($\frac{1}{2}$ doz.), sweet-williams: ($\frac{1}{2}$ doz.), chrysanthemums: (single), polyanthuses: (single), mesembryanthemums: ($\frac{1}{2}$ doz.), asters: ($\frac{1}{2}$ doz.))

$= \begin{pmatrix}210\\168\\88\end{pmatrix}$. His order £2·10, and neighbours £1·68 and 88p.

2. Cost per part matrix:

$$\begin{array}{c}\\\text{cost}\end{array}\begin{array}{c}\begin{array}{ccc}\text{Part 1}&\text{Part 2}&\text{Part 3}\end{array}\\\begin{pmatrix}3&2&4\end{pmatrix}\end{array}$$

Parts per subassembly matrix:

$$\begin{array}{c}\\\text{Part 1}\\\text{Part 2}\\\text{Part 3}\end{array}\begin{array}{c}\begin{array}{cc}\text{subassembly 1}&\text{subassembly 2}\end{array}\\\begin{pmatrix}3&2\\4&3\\3&5\end{pmatrix}\end{array}$$

Subassemblies per model matrix:

$$\begin{array}{c}\\\text{subassembly 1}\\\text{subassembly 2}\end{array}\begin{array}{c}\begin{array}{ccc}\text{Model A}&\text{Model B}&\text{Model C}\end{array}\\\begin{pmatrix}3&2&3\\3&4&5\end{pmatrix}\end{array}$$

Models per day matrix:

$$\begin{array}{c}\\\text{Model A}\\\text{Model B}\\\text{Model C}\end{array}\begin{array}{c}\text{day}\\\begin{pmatrix}8\\5\\4\end{pmatrix}\end{array}$$

Parts per model matrix:

$$\begin{array}{c} \\ \text{Part 1} \\ \text{Part 2} \\ \text{Part 3} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \text{subassembly 1} & \text{subassembly 2} \end{array} \\ \begin{pmatrix} 3 & 2 \\ 4 & 3 \\ 3 & 5 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ \text{subassembly 1} \\ \text{subassembly 2} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{Model A} & \text{Model B} & \text{Model C} \end{array} \\ \begin{pmatrix} 3 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ \text{Part 1} \\ \text{Part 2} \\ \text{Part 3} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{Model A} & \text{Model B} & \text{Model C} \end{array} \\ \begin{pmatrix} 15 & 14 & 19 \\ 21 & 20 & 27 \\ 24 & 26 & 34 \end{pmatrix} \end{array}$$

$$\text{Cost} = (3 \quad 2 \quad 4)\begin{pmatrix} 3 & 2 \\ 4 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \\ 4 \end{pmatrix} = (3 \quad 2 \quad 4)\begin{pmatrix} 15 & 14 & 19 \\ 21 & 20 & 27 \\ 24 & 26 & 34 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \\ 4 \end{pmatrix}$$

$$= (183 \quad 186 \quad 247)\begin{pmatrix} 8 \\ 5 \\ 4 \end{pmatrix} = 3{,}382 \text{ units.}$$

3. Contents matrix:

	wafers	whizzos	chocolate bars	toffees	sherbert	cigarettes	pipes	coins	balloons	packaging
"Sleigh Ride"	2	1	3	1	1	0	0	0	3	1
"Santa Claus"	1	1	1	1	1	1	1	0	6	1
"Jingle Bells"	0	1	4	0	0	2	2	6	12	1

Price matrix:

	price
wafer	1
whizzo	3
chocolate bar	2
toffees	4
sherbert	1
cigarettes	2
pipe	1
coin	1
balloons	$1\frac{1}{2}$
packaging	5

Order matrix:

Sleigh Ride	Santa Claus	Jingle Bells
10	15	10

$$\text{Cost of order} = (10 \quad 15 \quad 10)\begin{pmatrix} 2 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 6 & 1 \\ 0 & 1 & 4 & 0 & 0 & 2 & 2 & 6 & 12 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1\frac{1}{2} \\ 5 \end{pmatrix} = £11{\cdot}35.$$

page 26 $\mathbf{Mq} = \begin{pmatrix} 17 \\ 26 \\ 49 \\ 15 \\ 9 \end{pmatrix}$ the number of workers with each type of salary.

$\mathbf{pMq} = 116$, the total number of workers in the factory.

EXERCISE SET 7

1. (a) $\mathbf{nM}$ where $\mathbf{n} = (1 \ 1 \ 0 \ 0 \ 0)$.

(b) $\mathbf{Mm}$ where $\mathbf{m} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(c) $\mathbf{pMq}$ where $\mathbf{p} = (0 \ 1 \ 1 \ 1 \ 0)$; $\mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(d) $\mathbf{rMs}$ where $\mathbf{r} = (0 \ 0 \ 0 \ 1 \ 1)$; $\mathbf{s} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2. (a) $\mathbf{mN} = (0 \ 5 \ 4 \ 0 \ 1)$—members of class who have one sister.
$\mathbf{Nn} = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ —members who have two brothers.

$\mathbf{mNn} = 4$—members who have one sister and two brothers.

(b) (i) $\mathbf{pN}$ (ii) $\mathbf{Nq}$ (iii) $\mathbf{pNq}$.

(c) $\mathbf{rN} = (10 \ 12 \ 6 \ 0 \ 1)$—number of sisters for each number of brother.
$\mathbf{Ns} = \begin{pmatrix} 6 \\ 17 \\ 4 \\ 1 \\ 0 \end{pmatrix}$ —number of brothers for each number of sister.

(d) $\frac{29}{36} \simeq 1$—average number of sisters for each member of class.

(e) $\dfrac{\mathbf{pNs}}{\mathbf{pNq}}$

EXERCISE SET 8

1. Only in (b).

2. $a = 4$; $b = \frac{1}{2}$; $c = 2$; $d = -2$.

3. (a) $\begin{pmatrix}2 & 7\\1 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$ (b) $\begin{pmatrix}3 & -2\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}14\\6\end{pmatrix}$

(c) $\begin{pmatrix}3 & 2\\4 & 5\end{pmatrix}\begin{pmatrix}P\\Q\end{pmatrix} = \begin{pmatrix}14\\-11\end{pmatrix}$

page 31 Identify matrix **I**, when $\mathbf{I}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

EXERCISE SET 9

1. (a) $\mathbf{A} = \begin{pmatrix}2 & 7\\1 & 4\end{pmatrix}$

Let $\mathbf{E}_1 = \begin{pmatrix}\frac{1}{2} & 0\\0 & 1\end{pmatrix}$, then $\mathbf{E}_1\mathbf{A} = \begin{pmatrix}\frac{1}{2} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2 & 7\\1 & 4\end{pmatrix} = \begin{pmatrix}1 & \frac{7}{2}\\1 & 4\end{pmatrix}$

$\mathbf{E}_2 = \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}$; $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{7}{2}\\1 & 4\end{pmatrix} = \begin{pmatrix}1 & \frac{7}{2}\\0 & \frac{1}{2}\end{pmatrix}$

$\mathbf{E}_3 = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$; $\mathbf{E}_3(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & \frac{7}{2}\\0 & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}1 & \frac{7}{2}\\0 & 1\end{pmatrix}$.

$\mathbf{E}_4 = \begin{pmatrix}1 & -\frac{7}{2}\\0 & 1\end{pmatrix}$; $\mathbf{E}_4(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & -\frac{7}{2}\\0 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{7}{2}\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

Hence $\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$

and $\mathbf{A}^{-1} = \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \begin{pmatrix}1 & -\frac{7}{2}\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\0 & 1\end{pmatrix}$

$= \begin{pmatrix}1 & -7\\0 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\-\frac{1}{2} & 1\end{pmatrix}$

$= \begin{pmatrix}4 & -7\\-1 & 2\end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix}3 & 2\\1 & 1\end{pmatrix}$

Let $\mathbf{E}_1 = \begin{pmatrix}\frac{1}{3} & 0\\0 & 1\end{pmatrix}$, then $\mathbf{E}_1\mathbf{A} = \begin{pmatrix}1 & \frac{2}{3}\\1 & 1\end{pmatrix}$

$\mathbf{E}_2 = \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}$, $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & \frac{2}{3}\\0 & \frac{1}{3}\end{pmatrix}$

$\mathbf{E}_3 = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$, $\mathbf{E}_3(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & \frac{2}{3}\\0 & 1\end{pmatrix}$

$\mathbf{E}_4 = \begin{pmatrix}1 & -\frac{2}{3}\\0 & 1\end{pmatrix}$, $\mathbf{E}_4(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\mathbf{A}^{-1} = \begin{pmatrix}1 & -2\\-1 & 3\end{pmatrix}$

The pattern in finding the inverse of the matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ is to interchange the entries of the leading diagonal and change the sign of the entries in the other diagonal; $\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$

Inverse of $\begin{pmatrix}5 & -3\\2 & -1\end{pmatrix}$ is $\begin{pmatrix}-1 & 3\\-2 & 5\end{pmatrix}$ and $\begin{pmatrix}-1 & 3\\-2 & 5\end{pmatrix}\begin{pmatrix}5 & -3\\2 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

If $\mathbf{A} = \begin{pmatrix}4 & 2\\1 & 3\end{pmatrix}$, then the pattern gives $\mathbf{A}^{-1} = \begin{pmatrix}3 & -2\\-1 & 4\end{pmatrix}$; but $\begin{pmatrix}3 & -2\\-1 & 4\end{pmatrix}\begin{pmatrix}4 & 2\\1 & 3\end{pmatrix}$

$= \begin{pmatrix}10 & 0\\0 & 10\end{pmatrix} \neq \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

By the **E** matrices, $\mathbf{A}^{-1} = \begin{pmatrix}\frac{3}{10} & -\frac{1}{5}\\-\frac{1}{10} & \frac{2}{5}\end{pmatrix}$ and

$\begin{pmatrix}\frac{3}{10} & -\frac{1}{5}\\-\frac{1}{10} & \frac{2}{5}\end{pmatrix}\begin{pmatrix}4 & 2\\1 & 3\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

The factor 10 is explained in the next example.

2. $\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$

$\mathbf{E}_1 = \begin{pmatrix}\frac{1}{a} & 0\\0 & 1\end{pmatrix}$; then $\mathbf{E}_1\mathbf{A} = \begin{pmatrix}1 & \frac{b}{a}\\c & d\end{pmatrix}$

$\mathbf{E}_2 = \begin{pmatrix}1 & 0\\-c & 1\end{pmatrix}$; $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & 0\\-c & 1\end{pmatrix}\begin{pmatrix}1 & \frac{b}{a}\\c & d\end{pmatrix} = \begin{pmatrix}1 & \frac{b}{a}\\0 & -\frac{bc}{a} + d\end{pmatrix}$

$= \begin{pmatrix}1 & \frac{b}{a}\\0 & \frac{ad - bc}{a}\end{pmatrix}$

$\mathbf{E}_3 = \begin{pmatrix}1 & 0\\0 & \frac{a}{ad - bc}\end{pmatrix}$; $\mathbf{E}_3(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & 0\\0 & \frac{a}{ad - bc}\end{pmatrix}\begin{pmatrix}1 & \frac{b}{a}\\0 & \frac{ad - bc}{a}\end{pmatrix} = \begin{pmatrix}1 & \frac{b}{a}\\0 & 1\end{pmatrix}$

$\mathbf{E}_4 = \begin{pmatrix}1 & -\frac{b}{a}\\0 & 1\end{pmatrix}$; $\mathbf{E}_4(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix}1 & -\frac{b}{a}\\0 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{b}{a}\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

Hence $\mathbf{A}^{-1} = \mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \begin{pmatrix}1 & -\frac{b}{a}\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \frac{a}{ad - bc}\end{pmatrix}\begin{pmatrix}1 & 0\\-c & 1\end{pmatrix}\begin{pmatrix}\frac{1}{a} & 0\\0 & 1\end{pmatrix}$

$= \begin{pmatrix}1 & \frac{-b}{ad - bc}\\0 & \frac{a}{ad - bc}\end{pmatrix}\begin{pmatrix}\frac{1}{a} & 0\\-\frac{c}{a} & 1\end{pmatrix}$

$= \begin{pmatrix}\frac{1}{a} + \frac{bc}{a(ad - bc)} & \frac{-b}{ad - bc}\\-\frac{c}{ad - bc} & \frac{a}{ad - bc}\end{pmatrix}$ $\frac{1}{a} + \frac{bc}{a(ad - bc)} = \frac{ad - bc + bc}{a(ad - bc)} = \frac{d}{ad - bc}$

$= \begin{pmatrix}\frac{d}{ad - bc} & \frac{-b}{ad - bc}\\-\frac{c}{ad - bc} & \frac{a}{ad - bc}\end{pmatrix}$

$= \frac{1}{ad - bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$

3. (a) determinant $= 15 - 8 = 7$

inverse $= \frac{1}{7}\begin{pmatrix}5 & -2\\-4 & 3\end{pmatrix}$ $\frac{1}{7}\begin{pmatrix}5 & -2\\-4 & 3\end{pmatrix}\begin{pmatrix}3 & 2\\4 & 5\end{pmatrix} = \frac{1}{7}\begin{pmatrix}7 & 0\\0 & 7\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

(b) determinant $= -4 + 6 = 2$ (c) determinant $= 3 - 0 = 3$

inverse $= \frac{1}{2}\begin{pmatrix}-4 & -2\\3 & 1\end{pmatrix}$ inverse $= \frac{1}{3}\begin{pmatrix}3 & 0\\2 & 1\end{pmatrix}$

EXERCISE SET 10

1. (a) $\begin{pmatrix}2 & 7\\1 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4 & -7\\-1 & 2\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$$

$x = 5$ and $y = -1$.

(b) $\begin{pmatrix}3 & -2\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}14\\6\end{pmatrix}$ determinant $= -3 + 2 = -1$

$$\begin{pmatrix}a\\b\end{pmatrix} = -1\begin{pmatrix}-1 & 2\\-1 & 3\end{pmatrix}\begin{pmatrix}14\\6\end{pmatrix} = -1\begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}2\\-4\end{pmatrix}$$

$a = 2$ and $b = -4$

(c) $P = 6$, $Q = -7$.

2. $\mathbf{A} = \begin{pmatrix}5 & 2\\4 & 1\end{pmatrix}$; $\mathbf{A}^{-1} = -\frac{1}{3}\begin{pmatrix}1 & -2\\-4 & 5\end{pmatrix}$

(a) $x = 1, y = -2$; (b) $x = -7, y = 20$; (c) $x = -1, y = 4$.

3. $\begin{pmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{pmatrix}\begin{pmatrix}1 & -3 & 2\\-3 & 3 & -1\\2 & -1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

$$\begin{pmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 & -3 & 2\\-3 & 3 & -1\\2 & -1 & 0\end{pmatrix}\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$$

Hence $x = 1, y = -2$ and $z = 2$.

page 38 (1, −2); (−4, 3)
(−1, 2); (4, −3).

page 39 rotation through 90°, $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$; (−2, 0); (−1, −1).

rotation through 180°, $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$; (0, −2); (1, −1).

rotation through $-\alpha$, $\begin{pmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{pmatrix}$

page 41 $\begin{pmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 1\end{pmatrix}$; (0, 2).

EXERCISE SET 11

1. Reflection in Oy followed by rotation through 45° = (matrix rotation 45°)(matrix reflection)

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\-1 & 1\end{pmatrix} \qquad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

2. $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ = rotation through 270°.

3. $\begin{pmatrix}\cos\beta & -\sin\beta\\\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{pmatrix}$

$$= \begin{pmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\sin\beta - \cos\alpha\sin\beta\\\cos\alpha\sin\beta + \sin\alpha\cos\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\\sin(\alpha+\beta) & \cos(\alpha+\beta)\end{pmatrix}$$

Hence, equating corresponding entries

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

4. $\begin{pmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{pmatrix}$

$$= \begin{pmatrix}\cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha\\2\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha\end{pmatrix} = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\\sin 2\alpha & -\cos 2\alpha\end{pmatrix}$$

For the point $\begin{pmatrix}1\\2\end{pmatrix}$ in the line $y = x\tan 30°$ $\begin{pmatrix}\cos 60 & \sin 60\\\sin 60 & -\cos 60\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$

$$= \begin{pmatrix}\frac{1}{2} + \sqrt{3}\\\frac{\sqrt{3}}{2} - 1\end{pmatrix} = \begin{pmatrix}2{\cdot}23\\-0{\cdot}13\end{pmatrix}$$

5. $\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$$\begin{pmatrix}x''\\y''\end{pmatrix} = \begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$$

$x'' = p(ax + by) + q(cx + dy) = x(pa + qc) + y(pb + qd)$
$y'' = r(ax + by) + s(cx + dy) = x(ra + sc) + y(rb + sd)$

$$\begin{pmatrix}x''\\y''\end{pmatrix} = \begin{pmatrix}pa + qc & pb + qd\\ra + sc & rb + sd\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Hence $\begin{pmatrix}x''\\y''\end{pmatrix} = \begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}pa + qc & pb + qd\\ra + sc & rb + sd\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

and $\begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}pa + qc & pb + qd\\ra + sc & rb + sd\end{pmatrix}$

giving the row on column rule for matrix multiplication.

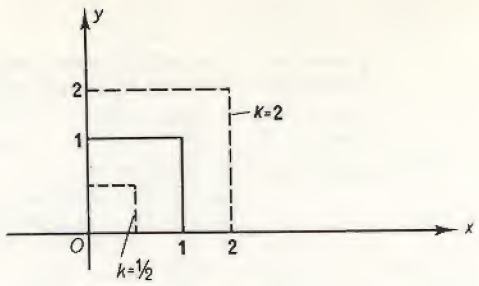
6. $\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\0 & 0 & 1\end{pmatrix}$

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 3\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2\sqrt{2}\\0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2\sqrt{2}\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{7\sqrt{2}}{2}\\1\end{pmatrix} = \begin{pmatrix}\cdot 71\\4{\cdot}95\\1\end{pmatrix}$$

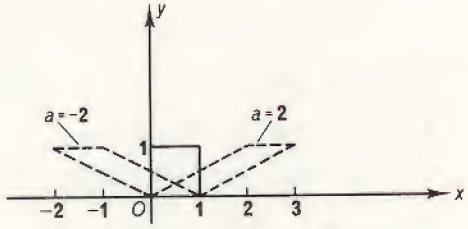
page 46

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$; $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

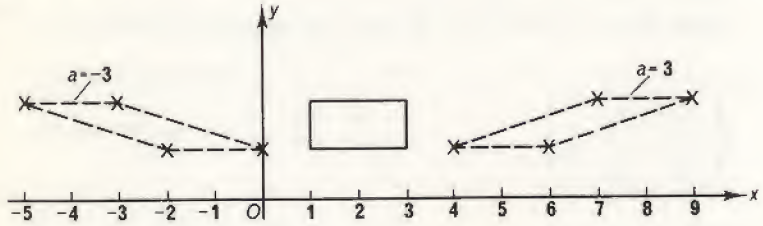


page 47 The matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ has the geometrical property of a linear stretch factor k parallel to Oy

page 48 $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$



$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$



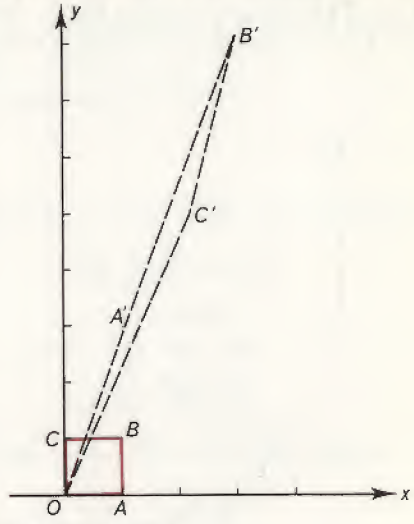
$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$—shear parallel to Oy.

page 50 $\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$

page 52

$$\begin{matrix} & O & A & B & C \end{matrix} \quad \begin{matrix} & O & A' & B' & C' \end{matrix}$$
$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 3 & 8 & 5 \end{pmatrix}$$

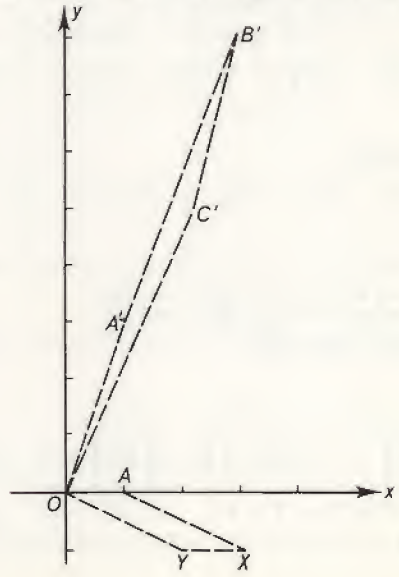
Note the interchange of sides for A and C with their maps A' and C'. A reflection is in the transformation of **A**, spotted by a negative valued determinant (-1).



$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and makes no change.

$\mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ and transforms $OA'B'C'$ as

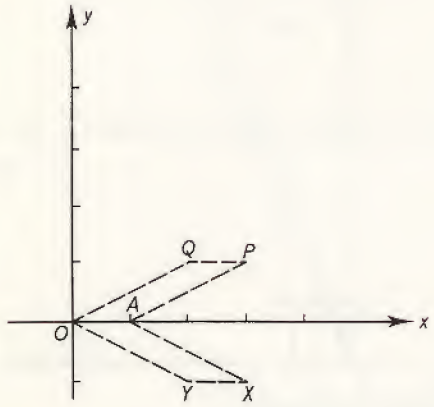
$$\begin{matrix} & O & A' & B' & C' \end{matrix} \quad \begin{matrix} & O & A & X & Y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 3 & 8 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$



$\mathbf{E}_2$ is the inverse of a shear matrix which has placed OA correctly, but $OAXY$ is beneath Ox.

Let $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$—a reflection in Ox.

$$\begin{matrix} & O & A & X & Y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{matrix} O & A & P & Q \\ \begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$



$\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as no stretch is required

$\mathbf{E}_4 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and transforms $OAPQ$ as

$$\begin{matrix} & O & A & P & Q \end{matrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{matrix} O & A & B & C \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

Hence $\mathbf{A}^{-1} = \mathbf{E}_4\mathbf{E}_3\mathbf{M}\mathbf{E}_2\mathbf{E}_1$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

EXERCISE SET 12

1.
$$\begin{matrix} & X & Y \end{matrix}$$
$$\begin{matrix} R \\ W \end{matrix}\begin{pmatrix} \frac{2}{5} & \frac{3}{4} \\ \frac{3}{5} & \frac{1}{4} \end{pmatrix}\begin{matrix} X \\ Y \end{matrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{matrix} R \\ W \end{matrix}\begin{pmatrix} \frac{2}{5} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2} \\ \frac{3}{5} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} \end{pmatrix} = \begin{matrix} R \\ W \end{matrix}\begin{pmatrix} \frac{23}{40} \\ \frac{17}{40} \end{pmatrix}$$

Probability Red marble drawn is $\frac{23}{40}$.

2.
$$\begin{matrix} & A & B \end{matrix}$$
$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{matrix} A \\ B \end{matrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\begin{pmatrix} \frac{1}{3} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{2} \\ 0 \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{2} \\ \frac{1}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\ \frac{1}{6} \times \frac{1}{2} + 0 \times \frac{1}{2} \\ \frac{1}{6} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{2} \end{pmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{12} \\ \frac{5}{12} \\ \frac{1}{12} \\ \frac{1}{6} \end{pmatrix}$$

Probability of a 1 is $\frac{1}{4}$, a 2 is $\frac{1}{12}$, a 3 is $\frac{5}{12}$, a 4 is $\frac{1}{12}$ and a 5 is $\frac{1}{6}$

3.
$$\begin{matrix} & \text{Up} & \text{Down} \end{matrix}$$
$$15\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{matrix} \text{Up} \\ \text{Down} \end{matrix}\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = 15\begin{pmatrix} \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{2}{3} \\ \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{2}{3} \end{pmatrix} = 15\begin{pmatrix} \frac{5}{6} \\ \frac{1}{6} \end{pmatrix}$$

A probability of $\frac{5}{6}$ I take the 15 bus.

4.
$$\begin{matrix} & A & B & C \end{matrix}$$
$$\begin{matrix} \text{Red} \\ \text{White} \\ \text{Black} \end{matrix}\begin{pmatrix} \frac{2}{5} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{3} & 0 \\ \frac{2}{5} & \frac{1}{3} & \frac{3}{5} \end{pmatrix}\begin{matrix} A \\ B \\ C \end{matrix}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{matrix} \text{Red} \\ \text{White} \\ \text{Black} \end{matrix}\begin{pmatrix} \frac{2}{5} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{2}{5} \times \frac{1}{3} \\ \frac{1}{5} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} + 0 \times \frac{1}{3} \\ \frac{2}{5} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{3}{5} \times \frac{1}{3} \end{pmatrix} = \begin{matrix} \text{Red} \\ \text{White} \\ \text{Black} \end{matrix}\begin{pmatrix} \frac{1}{3} \\ \frac{4}{15} \\ \frac{2}{5} \end{pmatrix}$$

The probability of drawing a black marble is $\frac{2}{5}$.

EXERCISE SET 13

1. (a) Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be an eigenvector of the matrix $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$ with eigenvalue λ,

then
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

so
$$\tfrac{1}{3}x + \tfrac{1}{2}y = \lambda x$$

and
$$\tfrac{2}{3}x + \tfrac{1}{2}y = \lambda y.$$

Then
$$x(\tfrac{1}{3} - \lambda) + \tfrac{1}{2}y = 0 \qquad (1)$$
$$\tfrac{2}{3}x + y(\tfrac{1}{2} - \lambda) = 0 \qquad (2)$$

For the gradients of these lines to be equal

$$\frac{-(\frac{1}{3} - \lambda)}{\frac{1}{2}} = \frac{\frac{2}{3}}{-(\frac{1}{2} - \lambda)}$$

Then
$$(\tfrac{1}{3} - \lambda)(\tfrac{1}{2} - \lambda) = \tfrac{1}{3},$$
$$\lambda^2 - \tfrac{5}{6}\lambda - \tfrac{1}{6} = 0,$$
$$6\lambda^2 - 5\lambda - 1 = 0$$
$$(6\lambda + 1)(\lambda - 1) = 0$$

and $\lambda = 1$ or $-\frac{1}{6}$.

Putting $\lambda = 1$ in (1) gives $-\frac{2}{3}x + \frac{1}{2}y = 0$ and in (2) $\frac{2}{3}x - \frac{1}{2}y = 0$; the same equation as $-\frac{2}{3}x + \frac{1}{2}y = 0$. Writing this equation as $4x = 3y$, the general solution is $x = 3t, y = 4t$. Hence the eigenvectors corresponding to the eigenvalue $\lambda = 1$, are $\begin{pmatrix} 3t \\ 4t \end{pmatrix}$. Putting $\lambda = -\frac{1}{6}$ in (1) gives $x + y = 0$. Hence the eigenvectors corresponding to the eigenvalue $\lambda = -\frac{1}{6}$ are $\begin{pmatrix} t \\ -t \end{pmatrix}$.

(b) $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

Hence $x(4 - \lambda) + 3y = 0$

$x + y(2 - \lambda) = 0.$

The characteristic equation for λ is $\lambda^2 - 6\lambda + 5 = 0$. Hence

$$(\lambda - 5)(\lambda - 1) = 0 \text{ and } \lambda = 1 \text{ or } 5.$$

The eigenvectors corresponding to $\lambda = 1$ are $\begin{pmatrix} t \\ -t \end{pmatrix}$ and corresponding to $\lambda = 5$, $\begin{pmatrix} 3t \\ t \end{pmatrix}$.

2. (a) $\frac{1}{3}$ (as there is a probability of $\frac{2}{3}$ he does not study).

(b) tomorrow
$$\begin{matrix} & S & NS \end{matrix} \quad \text{today}$$
$$\begin{matrix} S \\ NS \end{matrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$$

(c) 2nd day $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ probability of $\frac{1}{3}$ study 2nd day

3rd day $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{6} \\ \frac{5}{9} \end{pmatrix}$ probability of $\frac{4}{6}$ study 3rd day

4th day $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{4}{9} \\ \frac{5}{9} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\frac{5}{4} \\ \frac{3}{5}\frac{1}{4} \end{pmatrix}$ probability of $\frac{2}{5}\frac{3}{4}$ study 4th day.

(d) From **1**(a) above, the eigenvectors of the matrix $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$ corresponding to $\lambda = 1$, are $\begin{pmatrix} 3t \\ 4t \end{pmatrix}$ and since we seek the probability eigenvector such that $x + y = 1$, then $7t = 1$ and $t = \frac{1}{7}$, giving $x = \frac{3}{7}$, $y = \frac{4}{7}$. Hence on the last day of the endless term the probability the student is studying is $\frac{3}{7}$. The eigenvector corresponding to $\lambda = -\frac{1}{6}$ gives no solution with the requirement $x + y = 1$.

3. The transition matrix is

$$\begin{array}{cc} & \begin{matrix} \text{L} & \text{OT} \end{matrix} \quad \text{this lesson} \\ \text{next lesson} \begin{matrix} \text{L} \\ \text{OT} \end{matrix} & \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

The eigenvalues are 1 and $-\frac{1}{4}$.

The eigenvectors corresponding to eigenvalue $\lambda = 1$ are $\begin{pmatrix} 2t \\ 3t \end{pmatrix}$

The probability eigenvector is $\begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix}$ and hence there is a probability of $\frac{2}{5}$ the teacher is late for his lesson on the last day of term.

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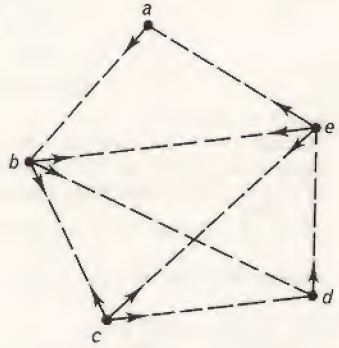
$$\mathbf{T} \star \mathbf{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$



$\mathbf{T} \star \mathbf{B}$ gives the routes between towns where a bus *or* a train route, but not both, exists.

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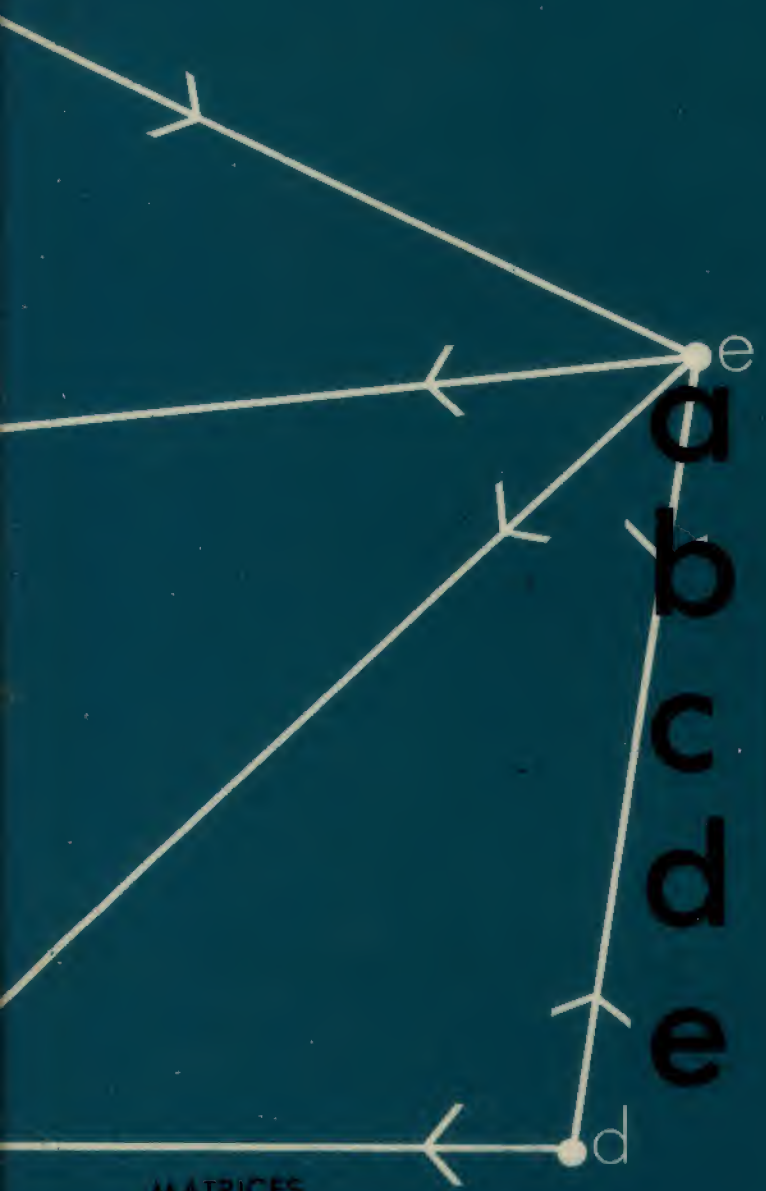

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